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A NOTE ON THE BENDING MOMENT INDUCED IN THE BOOMS OF A SPAR AT THE POINT OF APPLICATION OF A CONCENTRATED LOAD

by

H. F. WINNY, Ph.D., F.R.Ae.S.
(Head of Design, Fairey Aviation Co. Ltd.)

SUMMARY

The effect of a concentrated load, such as occurs in a wing spar at the fuselage, is to cause a discontinuity in shear strain of the spar web, which induces a bending moment in the booms through the medium of the rivets (or bolts) which attach the booms to the web. A mathematical theory is developed which shows that this bending induced in the boom is of a damped wave form starting at the point of application of the shear, and in practice the magnitude of the stresses produced by the bending moment may be appreciable at the wing root when the boom is deep compared with the spar depth (say 20 per cent.).

Further experimental verification is desirable, but a single wing test suggested the original investigation, and showed a measure of agreement with the theory.

1. INTRODUCTION

During a recent test to destruction on a complete wing, a failure occurred at the wing root on the bottom boom of the rear spar at a load appreciably less than that which would have been anticipated. The following investigation was undertaken to supply a possible explanation.

It is usual in spar design to provide spar booms to take the bending moment by end loads, and a thin web to take the shear. Where a sudden change in shear occurs, the change in web shear strain tends to give a sharp bend in the spar booms, which transfers some of the bending moment to the spar booms, each acting as separate beams and carrying the appropriate shears.

In the foregoing it has been assumed that the attachment between the web and the booms is capable of transmitting the vertical loads, and that the transmission

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is carried out with a corresponding deflection which, in the following analysis, will be assumed to be proportional to the load. This deflection will also be increased by the direct strain in the web, which may be estimated approximately by assuming that the vertical load acts on a vertical strip of web equal to one quarter of the depth of the web.

The effect of buckling of the spar web is ignored throughout the investigation, and where this occurs the results will be affected, but it is thought that the general tendencies will still persist.

Notation

S	shear relief carried by the boom
S_o	half the concentrated load
x	co-ordinate measured along the spar
y	upward deflection of the spar booms
E	modulus of elasticity
I	sectional moment of inertia of booms about their centroids
p	downward load per unit length of spar upper boom
y_w	deflection at centre line of web due to shear in the web
d	effective depth of web
N	shear modulus of elasticity
t	web thickness
k	constant of proportionality in equation (3)
A, B, C, D, C_1, C_2	constants in the solution of equation (5)
$\lambda_1, \lambda_2, \lambda_3, \lambda_4$	roots of equation (6)
y_o	the negative y co-ordinate of the intersection of the asymptotic to the deflected spar boom centre line at a great distance from the origin on the y axis (*see* Fig. 1)
α, β	real and imaginary parts of the roots of equation (6)
K	a non-dimensional parameter $(=\sqrt{(EIk)}/Ntd)$
M_{max}	maximum bending moment in spar boom
S_{max}	maximum shear carried by spar boom
p_{max}	maximum value of p

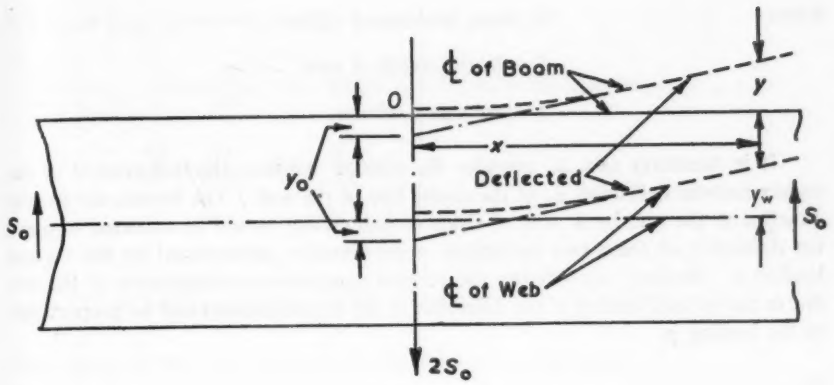

Fig. 1.

2. MATHEMATICAL ANALYSIS

Consider a uniform spar (Fig. 1) with a concentrated load $2S_o$ applied at a point O, and let O be taken as the origin for the co-ordinate x along the spar, and let y be the upward deflection of the spar booms from their original position. Since we are not concerned with the overall bending of the spar due to the extension and contraction of the spar booms, only the deflections due to shear of the web and the bending of the spar booms will be considered. The spar will be considered to be in equilibrium under the applied shear $2S_o$ and two opposite shears, S_o, acting at points $x \longrightarrow \pm\infty$.

For simplicity it will be assumed that the top and bottom booms are identical and each have a sectional moment of inertia about their respective centroids equal to I. Considering the bending deflection of the booms under a downward loading p per unit length,

$$EI\frac{d^4y}{dx^4} = -p, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where E = modulus of elasticity.

Considering the deflection y_w at the centre line of the web due to the shear in the web, and remembering that there are two booms to supply the loading,

$$\frac{d^2y_w}{dx^2} = -\frac{2p}{Ntd}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where N = shear modulus of rigidity

d = effective depth of web

t = spar web thickness.

It is necessary now to consider the relation between the deflection y of the booms and the deflection y_w of the centre line of the web. The booms are usually attached to the web by a row of rivets or bolts which would be expected to make the difference of these two deflections approximately proportional to the vertical loading p. Similarly, considering the vertical extension or compression of the web due to the vertical loading p, the difference of the two deflections will be proportional to the loading p.

Thus
$$k(y-y_w)=p, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where k is a constant of proportionality depending on the rivet attachment of the boom to the web and the web rigidity.

Eliminating p between (1) and (3)

$$y_w=y+\frac{EI}{k}\frac{d^4y}{dx^4}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and eliminating p between (2) and (3) and substituting for y_w from (4),

$$\frac{d^6y}{dx^6}-\frac{2k}{Ntd}\frac{d^4y}{dx^4}+\frac{k}{EI}\frac{d^2y}{dx^2}=0. \quad . \quad . \quad . \quad . \quad (5)$$

The general solution of (5) is

$$y=Ae^{\lambda_1 x}+Be^{\lambda_2 x}+Ce^{\lambda_3 x}+De^{\lambda_4 x}+C_1x+C_2,$$

where λ_1, λ_2, λ_3 and λ_4 are the four roots of the equation

$$\lambda^4-\frac{2k}{Ntd}\lambda^2+\frac{k}{EI}=0. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

To obtain the boundary conditions only positive values of x need be considered, and from symmetry, at the centre:

$$\frac{dy}{dx}=0 \quad \text{and} \quad \frac{d^3y}{dx^3}=0 \quad \text{when } x=0.$$

When x is large ($x \longrightarrow \infty$),

$$y = x\frac{S_o}{Ntd} - y_o, \quad \text{where } y_o \text{ is shown in the diagram,}$$

$$\frac{dy}{dx} = \frac{S_o}{Ntd} \quad \text{and} \quad \frac{d^2y}{dx^2} = 0.$$

Finally

$$y + y_o = \frac{S_o}{Ntd}\left[\frac{e^{-\alpha x}}{2(\alpha^2+\beta^2)}\{(3\alpha^2-\beta^2)\cos\beta x + (\alpha^2-3\beta^2)\sin\beta x\} + x\right],$$

where the roots for λ are expressed in the form $\pm\alpha \pm i\beta$ so that

$$\alpha = \sqrt{\left(\frac{Ntd}{2EI}\right)}\sqrt{(K+K^2)} \quad \text{and} \quad \beta = \sqrt{\left(\frac{Ntd}{2EI}\right)}\sqrt{(K-K^2)},$$

where $K = \sqrt{(EIk)}/Ntd$.

K is a non-dimensional parameter depending upon the three stiffnesses of the booms, the web and the effective attachment. It is seen that for all values of K (K must always be a real positive quantity) α is real, so that the oscillation is damped, but β changes from a real to an imaginary quantity as K becomes greater than unity. Thus the damped wave along the boom becomes "over-damped" as K becomes greater than unity.

It is now possible to write down expressions for the bending moment and shear in one of the booms, and the loading p on the boom.

$$\text{Bending moment in boom} = EI\frac{d^2y}{dx^2} = \frac{EIS_o}{2Ntd}(\alpha^2+\beta^2)e^{-\alpha x}\left(\frac{\cos\beta x}{\alpha} + \frac{\sin\beta x}{\beta}\right).$$

$$\text{Shear in boom} = EI\frac{d^3y}{dx^3} = -\frac{EIS_o}{2Ntd}\left(\frac{\alpha^2+\beta^2}{\alpha\beta}\right)e^{-\alpha x}\sin\beta x.$$

$$\text{Loading on boom} = -EI\frac{d^4y}{dx^4} = \frac{EIS_o}{2Ntd}e^{-\alpha x}(\alpha^2+\beta^2)\left(\frac{\cos\beta x}{\alpha} - \frac{\sin\beta x}{\beta}\right).$$

From the design view the maximum values of these quantities are of interest. The maximum bending moment occurs when $x=0$ and is shown to be equal to

$$M_{max} = \frac{EIS_o}{2Ntd}\frac{\alpha^2+\beta^2}{\beta} = S_o\sqrt{\left(\frac{EI}{Ntd}\right)}\frac{K}{\sqrt{(2+2K)}}.$$

The result is shown plotted against K in Fig. 2.

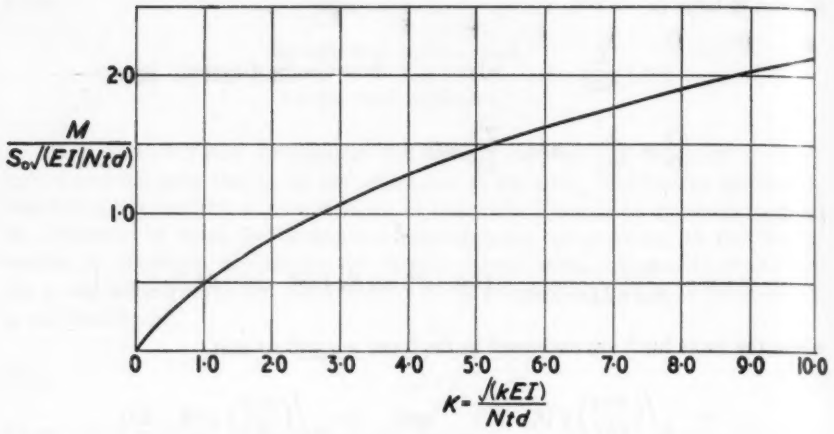

Fig. 2.

Maximum bending moment carried by boom.

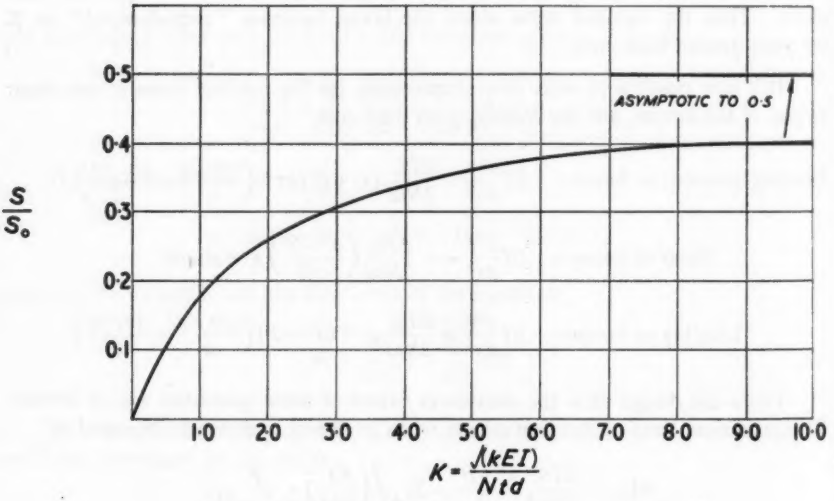

Fig. 3.

Maximum shear carried by boom.

The shear carried by the boom is a maximum when $\tan \beta x = \alpha/\beta$, and can be shown to be

$$S_{max} = S_o e^{-\alpha x} \frac{K}{\sqrt{(2+2K)}} = S_o \frac{K}{\sqrt{(2+2K)}} [K + \sqrt{(K^2-1)}]^{\sqrt{\left(\frac{K+1}{K-1}\right)}}$$

This shear S_{max} is shown plotted against K in Fig. 3. The shear is a relief to the main shear S_o carried by the web at a great distance from the applied shear $2S_o$. When K is less than unity a shear S exists which is additive to the shear S_o at a greater distance from the origin than S_{max}. This additive shear can be shown to be less than one per cent. of S_o.

The distance from the origin at which the maximum relief shear occurs is given by

$$x_o = \sqrt{\left(\frac{2EI}{Ntd}\right)\left(\tan^{-1}\sqrt{\left(\frac{1-K}{1+K}\right)}\right)} \Big/ \sqrt{(K-K^2)}$$

and is plotted in Fig. 5.

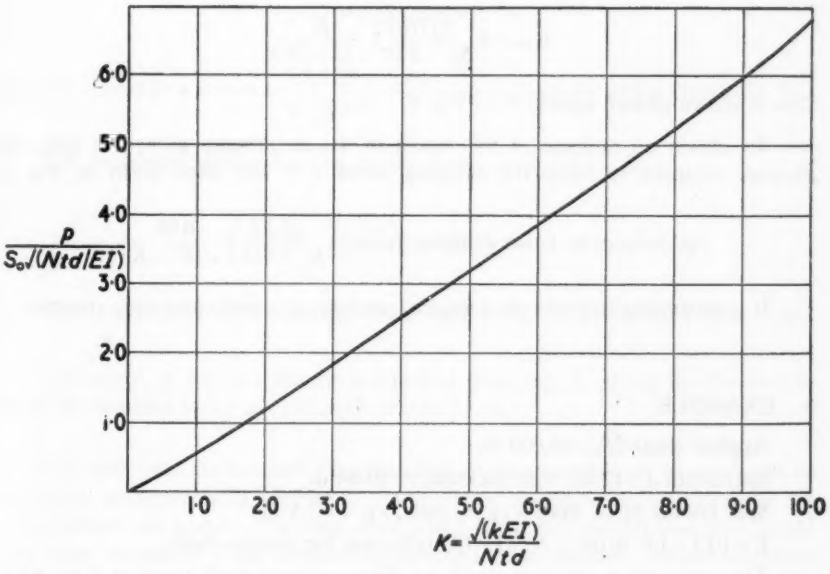

Fig. 4.
Maximum vertical loading on boom.

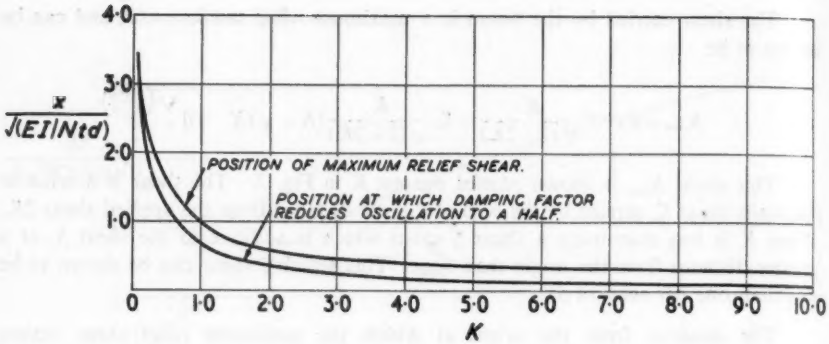

Fig. 5.

Position of maximum shear and half damping factor.

The loading on the attachment of the web to the boom is a maximum at the origin, and can readily be shown to be equal to

$$p_{max} = S_o \sqrt{\left(\frac{Ntd}{EI}\right)} \frac{K}{\sqrt{(2+2K)}}.$$

This is shown plotted against K in Fig. 4.

To obtain an estimate of the extent of the disturbance along the spar, the distance required to halve the damping factor $e^{-\alpha x}$ has been given in Fig. 5,

$$i.e.\ \text{distance to halve damping factor} = \sqrt{\left(\frac{EI}{Ntd}\right)} \frac{0.98}{\sqrt{(K^2+K)}}.$$

It is instructive to apply the foregoing analysis to a particular wing structure.

3. EXAMPLE

Applied shear $2S_o = 14{,}000$ lb.

Spar depth $d = 12$ in., web thickness $t = 0.064$ in.

Spar booms $3\frac{1}{8}$ in. deep $\times$ $1\frac{3}{8}$ in. wide, *i.e.* $I = 3.5$ in.^{4}

$E = 10.5 \times 10^6$ lb/in.^{2}; $N = 3 \times 10^6$ (allowing for tension field).

The spar web is attached by $\frac{3}{16}$ in. diameter snap head rivets at $\frac{1}{2}$ in. pitch.

The deflection at 1,000 lb./in. is approximately 0.003 in. and the extension of

the half depth of web, assuming an end load varying linearly from the rivet attachment to zero at the centre, is 0.0045 in. . Hence

$$k = \frac{1000}{0.003 + 0.0045} = 133{,}000 \text{ lb./in.}^2$$

The non-dimensional parameter $K = \sqrt{(kEI)/Ntd} = 0.96$.

This means that the disturbance is a damped wave, with the maximum boom bending moment, from Fig. 2,

$$M_{max} = 0.49 S_o \sqrt{\left(\frac{EI}{Ntd}\right)} = 13{,}800 \text{ lb. in.}$$

This corresponds to an additional stress on the outer fibres of the spar boom of 2.75 tons/in.^{2}:

The total relief shear in the web is, from Fig. 3,

$$S = 2 \times 0.18 S_o = 2{,}500 \text{ lb.}$$

and this occurs at a distance $\sqrt{\left(\frac{EI}{Ntd}\right)} \times 1.0 = 4.0$ in. from the origin (from Fig. 5).

The maximum vertical loading on the rivets, from Fig. 4, is

$$0.48 S_o \sqrt{\left(\frac{Ntd}{EI}\right)} = 840 \text{ lb./in.}$$

This is seen to be approximately 65 per cent. of the design strength of the rivets.

The extent of the disturbance is obtained from Fig. 5, giving the distance to halve the damping factor as $0.70\sqrt{(EI/Ntd)} = 2.8$ in.

It is seen from the example that the effect of the discontinuity in shear can be to produce an appreciable increase in the tensile or compressive direct stress in the outer fibres of the boom. The wing when tested failed at a smaller load than would have been estimated if the present analysis had been ignored. When allowance was made for the secondary bending effect the test failure was found to be in fair agreement with that predicted.

4. CONCLUSION

The present paper indicates that where a boom of a spar is deep compared with the depth of the spar (say over 20 per cent.) appreciable secondary bending moments may be produced by a large discontinuity in shear such as is likely to be present in a wing spar at the fuselage junction. Curves are given which enable this secondary bending moment to be estimated for design purposes together with the accompanying vertical loading on the rivets attaching the web to the boom, and also the relief shear in the web. The curves are plotted with a non-dimensional parameter K as the abscissa, this parameter depending upon the three stiffnesses of the spar booms, the web and the attachment of the web to the boom.

STRENGTH OF AEROPLANES IN RELATION TO REPEATED LOADS*

by

D. WILLIAMS, D.Sc., M.I.Mech.E., A.F.R.Ae.S.

(*Structures Department, Royal Aircraft Establishment*)

SUMMARY

The present position relating to the effect of repeated loads on aircraft structures is reviewed from the point of view of the capacity of aircraft structures to withstand such loads and also of their frequencies of occurrence in the life of the aircraft.

A rough account is given of the statistical methods by which such frequencies can be estimated, and the conclusions to which the use of these methods leads are discussed.

The note ends with an outline of a suggested programme of research and a brief statement of the conclusions reached.

1. INTRODUCTION

The somewhat mysterious failures of some Typhoon tailplanes during the war led to a fairly thorough investigation into the strength characteristics of their structure, including its behaviour under repetitions of loads well below the nominal failing load. In consequence of this latter work it was discovered, for example, that the successive application to 70 per cent. of the failing load could result in failure of the tailplane after about 1,000 repetitions. Further tests showed that failure at still lower loads could be produced by suitably increasing the number of repetitions.

At first sight these facts seemed to open up a somewhat disquieting prospect, for previously it had been tacitly assumed that failures under repeated loading were only to be expected as the result of fatigue, and fatigue was a phenomenon customarily associated with repetitions of the order of millions and hardly to be expected after a number to be measured in hundreds.

Further study of the loads which aeroplanes are likely to encounter in service

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* Modified Paper (A.R.C. No. 9965) received December 1948.

has indicated that the position is not nearly as serious as it was first thought to be, and that the prospect of having to test each new aeroplane not only to find its ultimate strength, but to find its behaviour under an elaborate programme of load repetitions at various fractions of that ultimate, may not after all be necessary.

In order to bring out this point more clearly, and for general convenience of treatment, it was thought desirable to divide this note into three sections, the first dealing with what we know of the effect of repeated loadings on actual aeroplane structures as the result of laboratory tests, the second dealing with what we know of the loads that actually occur in flight and the third outlining a programme of research designed to increase our existing knowledge in both these directions.

2. BEHAVIOUR OF STRUCTURES AND STRUCTURAL ELEMENTS UNDER REPEATED LOADINGS

Most of the standard data on repeated loading relates not to built-up structures nor even to structural elements but to simple specimens with a polished finish, designed to find the fatigue characteristics of the material itself. Recourse must therefore be had to the few tests that have been made recently, among which the Royal Aircraft Establishment tests on Typhoon tailplanes[1] and certain American tests[2] on riveted joints may be mentioned.

2.1.

To show the character of the results obtained, those relating to the Typhoon tailplanes are plotted in Fig. 1, which is self-explanatory. One notes immediately that there is some 10 per cent. scatter in the failing load of different specimens—a rather greater discrepancy than is usually to be expected, since similar tests on new Hudson tailplanes showed only 2 to 3 per cent. scatter for 10 specimens. This scatter makes it impossible to say that a particular specimen failed after repeated loading at a certain percentage of its own failing load, as the failing load cannot be defined except as a mean with a given deviation.

2.2.

Another point, and this has important consequences, is that even for repetitions of loads as high as 85 per cent. of the ultimate load we find no case when failure occurred after a very small number of repetitions—under twenty say.

It is safe to say that this is a general characteristic of all structures, for several striking examples have been obtained recently, among which may be mentioned the behaviour of the Armstrong Whitworth one-third scale wing specimen and that of the de Havilland Dove, in which loads very close to the breaking load were applied well over twenty times with no apparent ill effects. This point will be brought up again when dealing with loads to be expected in service.

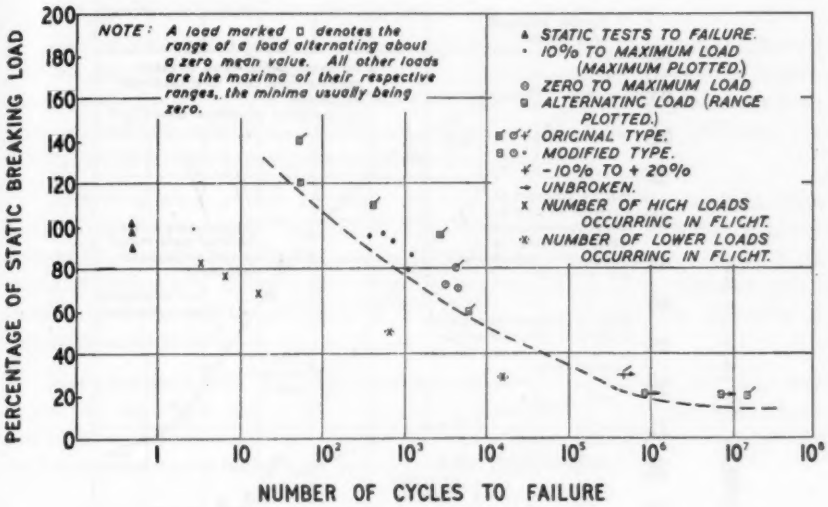

Fig. 1.

Fluctuating load tests of Typhoon tailplanes.

2.3.

A further point arising from Fig. 1 is that failure occurs much earlier under, for example, an alternating load of ± 60 per cent. of ultimate than under repetitions of loads from 0 to 80 per cent. This is in accordance with ordinary engineering experience relating to fatigue, and suggests that the phenomenon we are considering is not essentially different, being largely dependent upon the range of stress as well as the mean value.

2.4. *Stress raisers*

The question may be asked, how it is that repetitions at the lowest loads shown in the figure can produce failure, having regard to the fact that in standard fatigue tests the limiting fatigue stress as a fraction of the ultimate static load is much higher? The answer to this lies in the fact that, owing to the presence of regions of stress concentration, the actual stress is often many times higher than the nominal stress.

As is well known such stress raisers have little effect on the static failing load owing to plastic deformation relieving the high stress. This relief also helps the structure to withstand a much greater number of high load repetitions than it would otherwise. At low loads, however, the stress raisers are fully effective in reducing

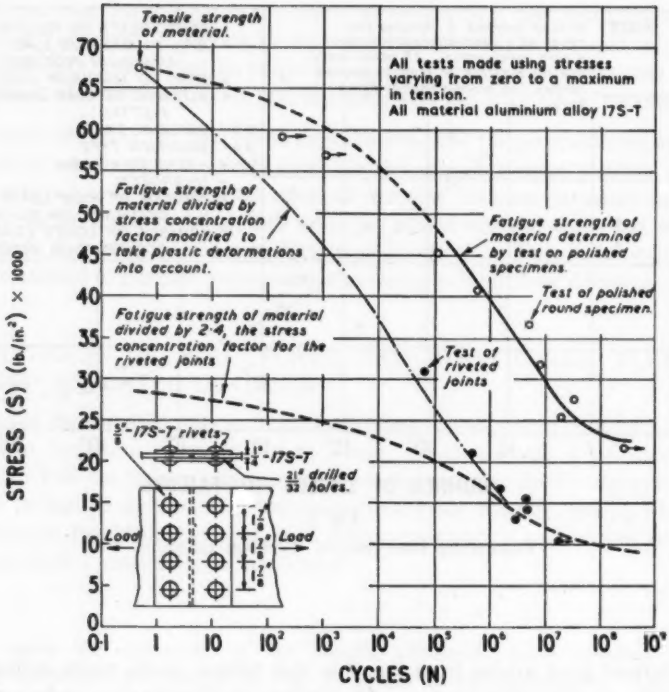

Fig. 2.

Direct tension fatigue tests on polished specimens and on riveted joints.

the fatigue strength, which then approaches that appropriate to the nominal stress concentration.

2.5.

This aspect of the matter is well discussed in an article, from which Fig. 2 is taken, by E. C. Hartmann[2] of the Aluminum Co. of America. Three *S-N* curves specimen, and the bottom curve—derived from the top curve by dividing the ordinates by 2.4, the estimated stress concentration—showing the estimated fatigue behaviour assuming the stress concentration were completely effective. The intermediate curve represents the actual behaviour and is based on the few experimental results available. At high values of *N* it is seen that the stress concentration *is* fully effective, but at low values the failing stress approaches that of the smooth test specimen, becoming equal to it at the static failing stress.

3. LOADS OCCURRING IN SERVICE

Data on loads occurring in service are derived mainly from *V-g* records collected over the past few years, largely on military aeroplanes during operational flights. Although conditions on civil air transports are in some respects very different from those on military machines, certain conclusions drawn from the collected *V-g* records can be accepted as equally applicable to civil machines.

Each *V-g* slide covers a period of 10 hours and records the maximum acceleration at every speed reached in flight over that period. Whereas it is not difficult to estimate the frequency with which the higher loads occur, there is no means of finding how frequent have been the smaller loads over the 10-hour period because of the complete smudging of the slide at levels below the silhouette of maximum accelerations.

Many hundreds of *V-g* records have been obtained over the past few years and in spite of the above limitations, a considerable amount of information has been derived from their examination. Before discussing this it will be useful if we divide the loads occurring in flight into three categories.

3.1. Types of repeated loads

The three types of repeated loads are the following:—

(*i*) Large loads of the order of half the ultimate load (or perhaps less) and upwards produced by exceptionally rough air conditions.

(*ii*) Smaller loads which are met constantly but intermittently under slightly bumpy air conditions.

(*iii*) Still smaller loads—many of them wholly negligible—produced by resonance of some part of the structure under the forcing frequencies of engine or airscrew. In contrast to (*i*) and (*ii*) these, if present at all, are always present.

It is worth while considering these three types of load carefully.

3.1.1. Loads of type (i)

An important aspect of this type of load is the fact that a good estimate of its frequency of occurrence in flight can be made from an examination of *V-g* slides—so long as there are enough of them. From a statistical point of view, the occurrence of such a load can be regarded as a rare event, and the chances of meeting it once, or twice, or any particular number of times in a given period can be estimated on the basis of the laws of probability.

The method is explained by Montagnon and Day[3], and they have established the fact that the occurrence of the exceptionally heavy flight loads does follow what is called the Poisson Distribution. This means that the chances of the particular

load not occurring at all, occurring exactly once, twice, thrice or any number of times r in a given period of time is given by the formula

$$e^{-m}\left(1, m, \frac{m^2}{2!}, \frac{m^3}{3!}, \ldots \frac{m^r}{r!} \ldots\right) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where m is the average number of occurrences to be expected in that period as deduced from the records.

It follows that while the chance of getting exactly r occurrences is $e^{-m}\frac{m^r}{r!}$, the chance of getting r *or more* occurrences is

$$\left[\sum_{s=r}^{\infty} \frac{m^s}{s!}\right] e^{-m}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in other words, the sum of that part of the series beyond and including $\left(\frac{m^r}{r!}\right) e^{-m}$.

If therefore r occurrences represent the safety limit, formula (2) represents the chance of an accident.

It is interesting to see how this works out in a particular instance. We shall take the case of the operational flight records of the Lancaster bomber for which some 700 slides have been recorded. An examination of these slides by Starkey and Cox[4] has shown that the occurrence of heavy loads (on a slightly pessimistic view) is as follows, assuming that the Lancaster is good for 4.75g at 60,000 lb.

3.25g (or 0.68 ultimate) once every 10^4 hours

3.63g (or 0.76 ultimate) once every 10^5 hours

3.91g (or 0.83 ultimate) once every 10^6 hours.

Suppose now we consider these figures applicable to the Lancaster as a civil aircraft—a severe assumption very probably—and, further, that we are catering for a service life of 50,000 hours, which is well above any so far recorded. Before we can decide how many times the Lancaster should stand the above loads on test we must decide what accident rate we are prepared to accept. As a tentative figure we may take one accident per 10^8 flying hours for any one aircraft, which means that the aircraft would have to repeat its life of 50,000 hours 2,000 times to produce a structural accident. The chance of an accident in the life of the aircraft, in other words, is one in 2,000; and the chance of its occurring during a 20-hour trip—a return trans-Atlantic flight for example—is consequently one in 5,000,000.

How small this chance is compared with the chance of an accident caused by

other and more serious air hazards may readily be gathered from the current passenger insurance rates for a return trans-Atlantic flight. This rate is about 0.02 per cent., and if we halve this so as to arrive at a figure that is more likely to represent the actual risk, we get the chance of a serious accident from all causes of one in 10,000—a five hundred times greater risk than the one in 5×10^6 mentioned before. There is clearly little point in reducing the structural accident risk below the suggested figure of one structural accident per 10^8 flying hours per aircraft, which therefore may be taken as meeting all reasonable safety requirements.

It is interesting to compare the above risks with motoring risks where the insurance rate is 0.04 per cent. per 400 hours roughly. Halving this rate to get a more realistic estimate of the actual risk gives 0.02 per cent. for 400 hours or 0.001 per cent. for a 20-hour trip. This is one in 100,000 and represents an accident risk 50 times greater than that chosen above for structural failure in the air.

Accepting the chance of structural accident in the life of the aircraft (50,000 hr.) to be one in 2,000, we have now to find what this means in terms of load repetitions at various high load levels, for no load must be repeated oftener in the life of the aircraft than will give a chance greater than one in 2,000 of producing failure. At each load level we require therefore to find the number of repetitions whose chance of being reached (or exceeded) is just one in 2,000. Taking for example, a load of $3.25g$, we note that on the average it occurs once in every 10^4 hours or 5 times in 50,000 hrs. The value of m in the Poisson distribution is therefore 5 and we have to find the value of r such that

$$\sum_{r}^{\infty} e^{-m} \frac{m^r}{r!} = \frac{1}{2{,}000} \quad . \quad . \quad . \quad . \quad (3)$$

This gives $r = 15$, which means that if we apply a load of $3.25g$ (*i.e.* 0.68 ultimate or proof load approximately) 15 times to the structure we are ensuring an accident rate of one in 10^8 hours. Should even that rate be considered too high it is interesting to note that by making the number of repetitions 17 instead of 15 the accident rate is 10 times smaller, *i.e.* one in 10^9 hours.

Consider the next load of $3.63g$ (0.76 ultimate); its average occurrence in the life of the aeroplane is $50{,}000/10^5 = 0.5$ times and the number of repetitions whose chance of being exceeded is $1/2{,}000$ is again given by equation (3) with m this time equal to 0.5. We find now that, for an accident rate of one in 10^8 hours, the number of repetitions is 5, and for a rate of one in 10^9 hours the number is 6.

The number of repetitions for the load of $3.91g$ (0.83 ultimate) is still lower, being 3 both for the one in 10^8 hours accident rate and for the one in 10^9 hours rate.

It is interesting to note that as the load becomes higher, and therefore more and more of a rare event, the number of repetitions required to cover the stipulated

accident rate becomes increasingly greater when expressed as a multiple of the number directly deduced from the average rate of occurrence of the load and the life of the aeroplane. Thus we have

Load	3.25g	3.63g	3.91g
Average rate of occurrence	1 in 10^4 hr.	1 in 10^5 hr.	1 in 10^6 hr.
Hence average number of occurrences in 50,000 hr.	5	0.5	0.05
Number to be catered for to give 1/2,000 chance of occurring	15	5	3
Latter expressed as multiple of average number ...	3	10	60

Conversely, as the loads drop and their frequencies increase, the number of repetitions of a particular load to be catered for approaches ever nearer, that deduced directly from the average rate of occurrence as given by the appropriate frequency curve, until at such loads as 30 per cent. of ultimate (1.42g) they are practically equal.

Summarising the figures appropriate to loads in the first category we get the following table:—

Ultimate strength: 4.75g

Life of aircraft: 50,000 hours.

Load in terms of g	3.25	3.63	3.91
Load as fraction of ultimate	0.68	0.76	0.83
Repetitions for required degree of safety:			
case (*i*)*	15	5	3
case (*ii*)*	17	6	3

Assuming now that the strength of the structure can be represented, in so far as the relation between repeated loading strength and ultimate strength is concerned, by the broken curve of Fig. 1, it is instructive to plot the above "required" repetitions against the actual capacity of the structure. This has been done for case (*ii*) of the above table which is little different, as the table shows, from case (*i*). The three points are shown by crosses and a comparison of these "required"

* Case (*i*) relates to an accident rate of one in 10^8 hours.
(*ii*) relates to an accident rate of one in 10^9 hours.

repetitions with the number the structure can stand demonstrates their essential insignificance. It is also to be remembered that, small as are the "required" numbers of repetitions, they are still more than a civil aircraft need withstand, since the average frequency of occurrence has been based on war operational flights during which a greater frequency of high loads—the consequence of violent evasive actions etc.—are to be expected.

We are thus led to the important tentative conclusion that an aeroplane stands in no danger from the possibility of failure by the repeated application of large loads—meaning by large anything above half the ultimate. The conclusion must remain tentative naturally until it has been shown that other typical aeroplane structures are not less proof than the Typhoon tailplane against repeated loadings. In the writer's view however there is little doubt that this will be amply demonstrated. Tests carried out at R.A.E. since the Typhoon tailplane tests have shown results well in accord with this view.

3.1.2. Second category of loads

As the loads become smaller they naturally become more frequent and, *given their average rate of occurrence*, the task of estimating how many times they are likely to occur over a given period of time is more straightforward. It is intuitively clear for example that the greater the number of times an event occurs in a given time the more likely is that number to agree with the average rate of occurrence, and the smaller the chance of any appreciable deviation from it.

For example, still on the basis of the *V-g* records for the Lancaster, it has been estimated that the proper number of repetitions of a 50 per cent. load such that the chances of reaching that number in the life of the aircraft is one in 20,000 (*i.e.* using the accident rate of one in 10^9 hours) is 610. This figure is derived by first obtaining the figure of 500 directly from the average frequency given by the *V-g* records and then increasing it to 610 on the basis that the chance of exceeding this average by 22 per cent. (*i.e.* by 110) is one in 20,000. This well emphasises the fact that as the frequency of the load increases the number obtained by dividing the life of the aircraft by the average period between occurrences of the load becomes increasingly suitable as the number of repetitions to which the aircraft should be tested. This again leads to the conclusion that the obvious desiderata are more flight statistics of a kind where clear records of all loads, high and low, are obtained on an extended time base in order to establish a reliable average.

3.1.3. Third category of loads

This category includes the very low loads of high frequency induced by forced vibrations set up by engine or airscrew. The magnitude of these loads can be

directly measured for any particular aircraft and their number in any given period is immediately obtained by multiplying the frequency by the time. At 40 per sec., for example, in the course of a life of 50,000 hours the formidable total of some 7×10^9 repetitions is obtained. In many cases the stresses involved are so small as to make any effect on the structure negligible. In others, however, serious account has to be taken of them, the Typhoon tailplane being an example. At one time the resonance vibration induced in the tailplanes of these machines by engine forces was so heavy that nominal alternating stresses of some 3,000 lb./in.^{2} were produced, resulting in failure of either the spar root fittings or the half-inch root-holding bolts.

It may well be that resonance vibrations of this type are the most dangerous of all types of repeated loadings on aircraft, for although the alternating stresses are small in themselves, they are often superimposed on large locked-up stresses such as are found in tightened bolts, in spar webs with dished lightening holes and in welded joints.

In contrast, however, to the heavier aerodynamic loads, vibration stresses can be reduced at the source, for example by careful design of engine mountings, so that even if they are the most dangerous they are also the most tractable from the designer's point of view.

It is the author's view that every aeroplane should be thoroughly tested in flight so that the character of the vibrations present and the amplitudes of the stresses to which they give rise may be known with certainty, and so that steps may be taken to reduce such stresses if thought to be excessive.

4 VARIATION IN STRENGTH OF NOMINALLY IDENTICAL STRUCTURES

There is a further point which needs to be considered and which, so far, has only received casual mention in relation to the static failing load. This is the scatter of the points that give the curve of structural strength against number of load repetitions. The difficulty always arises in fatigue* tests that it is impossible to relate a load that produces fatigue failure to the static breaking load *for the same test specimen*: it can only be related to the average static breaking load of nominally identical structures. If, for example, the test specimen happens to be 5 per cent. below average in static strength and it is subjected to a nominal 20 per cent. fatigue load, it is in fact subjected to 21 per cent. of its own ultimate load—a difference that might well prolong its life by a good percentage in view of the small slope of the logarithmic *S-N* curve in the region of 20 per cent. of the ultimate load. This

* Dr. P. B. Walker in his paper "The Fatigue of Aircraft Structures"(5) has justified the substitution of the older term "fatigue" for the more clumsy term "repeated loading."

emphasises the vital need of knowing the distribution of the numbers of repetitions at which seemingly identical structures break under various loads. This need is clearly more urgent for the lower loads than for the high, because of the point just made and because the margin of safety for the high loads is, as already pointed out, never likely to be critical in any practical structure.

5. REASSURING FEATURES

It has been suggested that danger of fatigue failure is greater from the frequent occurrence of comparatively small loads up to 15 or 20 per cent. of the ultimate, than from the less frequent occurrence of higher loads, and that for such low loads the lives of nominally identical specimens are likely to show a wide scatter. Thus, whereas one specimen fatigue tested to 10 per cent of its ultimate load might fail after 2 million repetitions, another, seemingly identical, specimen might last for 2½ or 3 million repetitions. This simple fact has a very important bearing on the aircraft designer's attitude to the fatigue problem, for, properly exploited, it can relieve him from the fear of catastrophic structural failure from fatigue. The way to take advantage of this scatter is to ensure that all important loads in the aircraft structure shall be shared more or less equally among as large a number of equal elements as possible. Fortunately, this is just the condition that is being brought about by present trends in design; heavy spars are disappearing with their accompanying heavy joints, and the main loads in wings, fuselages, and tailplanes are being increasingly catered for by a multiplicity of stringers, with transverse joints made by a corresponding multiplicity of bolts. The big scatter—of the order of thousands of flight hours—among the lives of such individual elements ensures that inspection at reasonable intervals cannot fail to detect incipient damage. Any trouble from the stressed skin, divided as it is into individual spanwise strips by the reinforcing stringers, is easily spotted long before it can have serious consequences. The stringers themselves are not so easily inspected, but tests have shown that extensive damage to these cannot take place without betraying the fact through the formation of cracks in the adjacent skin. The point may be made that, if one or two stringers fail, the others cannot be in a condition safely to resist a sudden heavy load. This fortunately is not a real danger, for experience shows that static strength is little affected by previous fatigue history.

Admittedly the internal shear webs of a wing present a more difficult problem for the inspector, since initial damage to these may leave the skin intact. The only remedy is to design for easy inspection of the outer webs (for a multi-web wing), the condition of which may be taken as a fair guide to that of the others.

It is obvious that if incipient fatigue damage does occur the problem facing those responsible for structural maintenance is likely to be formidable. The heartening feature, however, is that lessons of the utmost value for future design of fatigue-free structures will have been learnt at no cost in lives.

6. SUGGESTIONS FOR FURTHER WORK

The suggestions put forward here follow naturally from what has been said already. We need

(1) To obtain more statistical flight records of an improved type so that the number of times accelerations greater than, for example $1g$, $1\frac{1}{8}g$, $1\frac{1}{4}g$, and so on, are encountered, can be counted.

The stress at spar roots could probably also be measured on a statistical basis, "statistical" referring to records obtained by the use of special instruments on civil air liners operating on standard routes.

(2) To obtain records of a statistical nature but on an experimental basis. By this is meant the temporary allocation of research personnel to make special investigations on selected routes. In the course of such investigations a certain degree of elaboration of equipment could be used to enable synchronised records of accelerations and stresses to be obtained. (Much work on these lines has been done by the Royal Aircraft Establishment in the course of the past year.)

(3) To obtain further data on the behaviour of structures and structural elements under repeated loading. The directions in which it is thought work could profitably be canalised are indicated below:—

(*i*) Work on high loads (50 per cent. of ultimate and above) applied to complete structures with particular reference to the determination of the degree of scatter at each of the various loads. Also it is desirable to determine the effect on the mean strength at high loads of applying the full quota of low loads first. High loads are specifically mentioned because it is hoped that this work will be comparatively short, the aim being to establish the truth of the proposition already made that such loads are not important.

(*ii*) Work on structural elements such as riveted joints with a view to finding their behaviour over the range terminated by the ultimate static load at the high end and the limiting fatigue loads at the lower end, thus defining more clearly the middle curve of Fig. 2. This work could include investigating the scatter of the number of repetitions to failure at each load, which means that at least 10 specimens should be tested at each load.

(*iii*) In the course of work on low loads the effect on the mean value and on the scatter should be investigated of changing the character of the load on identical structures and of applying similar loads to different structures. For example, a particular design of wing might be tested under bending and torsion, and different wing types might be compared under bending loads, or a wing and a fuselage.

(*iv*) Work on the effect of high frequency vibration on complete structures and on structural elements whose design favours the presence of locked-up stresses, the degree of scatter of the results receiving particular attention. *This item is considered to be the most important of all those mentioned here.*

A consequence of adopting this rough programme would be the necessity for making a large number of specimens of complete structures as well as of structural elements; for it is essential that every result should be presented as a mean with a certain well defined scatter.

7. CONCLUSIONS

The following is a brief statement of the conclusions to which the present review appears to lead.

(*i*) There is a fair prospect that the incidence of repeated or oscillating loads on aircraft, apart from stresses induced by high frequency resonance vibrations, will present no serious problem in the future. It is thought that if a structure satisfies the ultimate load test (and the factor of safety bound up with its present definition) after taking full account of the scatter, it will, barring radically bad design from a fatigue point of view, be amply strong enough to withstand any repeated loads it is likely to encounter in service.

(*ii*) Oscillating stresses produced by resonance vibration caused by engine or airscrew forces are potentially dangerous, as they are likely to be superimposed on existing locked-up stresses from which no aircraft structure can be entirely free. It is comforting to note however that it is in the designer's power to reduce such stresses to negligible proportions by the avoidance of designs favourable to locked-up stresses and by effectively insulating the airframe from the engine or, still better, by reducing the periodic forces at their source. In this connection jet propulsion offers probably a complete answer so far as airframe vibration stresses are concerned.

(*iii*) Work should now be directed towards establishing the view expressed in (*i*) by collecting the necessary statistical flight data and making the appropriate laboratory tests. Statistics directed to measuring the rate of occurrences of the stresses are irrelevant to item (*ii*), but a great deal of laboratory work on the effect of vibration stresses will have to be done.

(*iv*) The relative immunity from catastrophic fatigue failure attainable by a form of design (now fortunately current) in which important loads are shared more or less equally among a number of similar structural elements is a reassuring feature in the problem of aircraft fatigue.

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THE SUPERSONIC FLOW PAST A SLENDER DUCTED BODY OF REVOLUTION WITH AN ANNULAR INTAKE

by

G. N. WARD

(Department of Mathematics, The University, Manchester)

SUMMARY

The approximate supersonic flow past a slender ducted body of revolution having an annular intake is determined by using the Heaviside operational calculus applied to the linearised equation for the velocity potential. It is assumed that the external and internal flows are independent. The pressures on the body are integrated to find the drag, lift and moment coefficients of the external forces. The lift and moment coefficients have the same values as for a slender body of revolution without an intake, but the formula for the drag has extra terms given in equations (32) and (56). Under extra assumptions, the lift force due to the internal pressures is estimated. The results are applicable to propulsive ducts working under the specified condition of no " spill-over " at the intake.

1. INTRODUCTION

The problem of determining the approximate supersonic flow past a slender body of revolution whose meridian section has discontinuities of slope has been solved by Lighthill[1], who has given expressions for the drag coefficient at zero incidence and for the lift and moment coefficients at incidence. A ducted body with a circular intake at the nose is a special case in Lighthill's general results. In this paper the aerodynamic forces are found on a slender ducted body of revolution having an annular intake following a pointed fore-body. It is assumed that the fluid which enters the body through the intake continues to move at supersonic speed for at least a short distance inside the body, so that the external and internal flows are independent. This assumption is justified if the internal flow is not choked

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to such an extent that "spill-over" occurs at the intake, and if the effect of the boundary layer on the fore-body is not such as to cause instability of the flow. All the effects of this boundary layer are neglected here.

The isentropic flow of an inviscid gas past the body is considered, and it is assumed that the disturbance velocity caused in an initially uniform stream is sufficiently small for its square and higher powers to be neglected in the equation of motion. The disturbance velocity may then be determined from a potential function satisfying the Prandtl-Glauert equation. This assumption imposes the condition that the tangent plane to the surface of the body at any point must make a small angle (such that its square is negligible compared with unity) with the direction of the undisturbed stream. In order to calculate the pressures on the body, the gas is assumed to be perfect and to have constant specific heats.

Notation

A, A_0, A_1	arbitrary functions of p, determined from boundary conditions.
B	$\sqrt{(M^2-1)}$.
C, C_0, C_1	arbitrary functions of p, determined from boundary conditions.
D	drag force measured in direction of undisturbed stream.
$F(x)$	$\int_0^x dx \int_0^x S''(y)\,dy$.
$H(x)$	Heaviside unit function
K_0, K_1	modified Bessel functions of the second kind.
L, L_1	functions defined in equations (41) and (48).
M	Mach number.
M_0	moment of lifting forces about the origin.
N	force normal to body axis.
p	Heaviside operator, $1/p \equiv \int_0^x dx$.
$R(x)$	radius of body at distance x from nose.
R_0, R_1	radii of body just before and just after the annular intake, $R_0 = R(x_1 - 0)$, $R_1 = R(x_1 + 0)$.
r, θ, x	cylindrical polar co-ordinates.
$S(x) = \pi[R(x)]^2$	area of cross-section of body at distance x from nose.

t maximum thickness of body

U function defined in equation (23).

$\mathscr{U}$ velocity of undisturbed stream.

V function defined in equation (41).

v_r, v_θ, v_x radial, tangential and axial components of velocity.

x_1 distance of annular intake from nose of body.

α angle of incidence of body to undisturbed stream.

η_0, η_1 slope of meridian sections of body just before and just after the annular intake, $\eta_0 = R'(x_1 - 0)$, $\eta_1 = R'(x_1 + 0)$.

ϕ, ϕ_0, ϕ_1 potential functions: $\phi = \phi_0 + \phi_1$, ϕ_0 giving the flow at zero incidence, and ϕ_1 giving the extra disturbance due to incidence.

ρ density of undisturbed stream.

2. THE EQUATIONS OF MOTION AND THE BOUNDARY CONDITIONS

Let r, θ, x be a system of cylindrical co-ordinates such that the x-axis coincides with the axis of the body and the origin of co-ordinates is at the pointed nose. Let $\mathscr{U}$ be the velocity and M be the Mach number of the undisturbed stream; let $\mathscr{U}\cos\alpha$ and $\mathscr{U}\sin\alpha$ be the components of the stream velocity parallel and normal to the x-axis respectively, so that α is the incidence of the body to the stream, and take $\theta = 0$ to be in the direction of the component $\mathscr{U}\sin\alpha$.

Since the flow is isentropic, and hence irrotational, the components of the velocity are given by

$$\left.\begin{aligned} v_r &= \mathscr{U}\,(\sin\alpha\cos\theta + \partial\phi/\partial r), \\ v_\theta &= \mathscr{U}\,(-\sin\alpha\sin\theta + \partial\phi/r\partial\theta), \\ v_x &= \mathscr{U}\,(\cos\alpha + \partial\phi/\partial x), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

where $\mathscr{U}\phi$ is the disturbance velocity potential.

If $R(x)$ is the radius of the body at any section $x = \text{constant}$, the boundary condition of zero normal velocity at the body surface is

$$v_r = R'(x)\, v_x. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(Dashes denote differentiation with respect to x).

The other boundary condition is that ϕ and all its derivatives must vanish at points upstream from the body.

Since the disturbance velocity is small and the body is slender, α and $R'(x)$ must be small, so that ϕ satisfies

$$\left.\begin{aligned} &\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=B^2\frac{\partial^2\phi}{\partial x^2}, \\ &B^2=M^2-1, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

and the boundary condition (2) becomes, approximately,

$$\alpha\cos\theta+\partial\phi/\partial r=R'(x). \quad . \quad . \quad . \quad . \quad (4)$$

Consider a body of unit length with an annular intake at the section $x=x_1$. Let $R(x)$ be analytic in $0\leqslant x\leqslant 1$ except at $x=x_1$, and let

$$R(x_1-0)=R_0,\ R(x_1+0)=R_1, \quad . \quad . \quad . \quad . \quad (4)$$

$$R'(x_1-0)=\eta_0,\ R'(x_1+0)=\eta_1, \quad . \quad . \quad . \quad . \quad (5)$$

$$S(x)=\pi[R(x)]^2=\text{area of cross-section.} \quad . \quad . \quad . \quad (6)$$

Then
$$S'(x)=F'(x)+2\pi(R_1\eta_1-R_0\eta_0)H(x-x_1), \quad . \quad . \quad . \quad (7)$$

$$S(x)=F(x)+\pi(R_1^2-R_0^2)H(x-x_1)+2\pi(R_1\eta_1-R_0\eta_0)(x-x_1)H(x-x_1), \quad (8)$$

where $H(x)$ is the Heaviside unit function,

$$H(x)=\begin{cases} 0 \text{ when } x<0 \\ 1 \text{ when } x>0, \end{cases}$$

and
$$F(x)=\int_0^x dx\int_0^x S''(y)\,dy$$

$F(x)$ is continuous together with its first derivative in $0<x<1$.

Since the disturbance vanishes for $x<0$, the Heaviside operational form of equation (3) is

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=B^2p^2\phi. \quad . \quad . \quad . \quad (9)$$

The potential ϕ may be split into two parts ϕ_0 and ϕ_1 satisfying respectively the two following conditions on the body derived from equation (4):—

$$\partial\phi_0/\partial r = R'(x), \text{ or } 2\pi r\partial\phi_0/\partial r = S'(x), \quad . \quad . \quad . \quad . \quad (10)$$

and

$$\partial\phi_1/\partial r = -\alpha\cos\theta, \text{ or } \pi r^2\partial\phi_1/\partial r = -\alpha\cos\theta\, S(x). \quad . \quad . \quad (11)$$

ϕ_0 gives the flow at zero incidence and ϕ_1 gives the extra disturbance due to the incidence. From the form of equations (10) and (11), the appropriate solutions of equation (9), representing waves progressing outwards and downstream, are

$$\phi_0 = A(p)\,K_0(Bpr). \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and

$$\phi_1 = C(p)\,K_1(Bpr)\cos\theta, \quad . \quad . \quad . \quad . \quad (13)$$

where $A(p)$ and $C(p)$ are functions of p to be determined from equations (10) and (11) respectively, and K_0, K_1 are modified Bessel functions.

3. THE FLOW AT ZERO INCIDENCE

When $x = x_1$, the streamlines through $r = R_1$ will change direction discontinuously, in general, and we are led to introduce into ϕ_0 a term of the form

$$-\frac{A_1K_0(Bpr)}{BpK_1(BpR_1)}e^{-px_1},$$

which will give a discontinuity A_1 in $\partial\phi_0/\partial r$ at $x = x_1$, $r = R_1$. The potential ϕ_0 then takes the form

$$\phi_0 = A_0(p)\,K_0(Bpr) - \frac{A_1K_0(Bpr)}{BpK_1(BpR_1)}e^{-px_1}, \quad . \quad . \quad . \quad (14)$$

so that

$$\frac{\partial\phi_0}{\partial r} = -BpA_0(p)\,K_1(Bpr) + \frac{A_1K_1(Bpr)}{K_1(BpR_1)}e^{-px_1}, \quad . \quad . \quad . \quad (15)$$

On and near the body, where r is small, the Bessel functions may be expanded in power series, and

$$r\partial\phi_0/\partial r = -A_0(p) + A_1R_1e^{-px_1}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

approximately. On the body,

$$2\pi r\partial\phi_0/\partial r = S'(x) = F'(x) + 2\pi(R_1\eta_1 - R_0\eta_0)H(x-x_1), \quad . \quad . \quad (17)$$

from equations (7) and (10), and so

$$A_0(p) = -\frac{1}{2\pi}F'(x), \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and

$$A_1 = (R_1\eta_1 - R_0\eta_0)/R_1. \quad . \quad . \quad . \quad . \quad (19)$$

Since $\phi_0 = 0$ when $x \leqslant 0$,

$$\frac{\partial\phi_0}{\partial x} = p\phi_0 = pA_0(p)K_0(Bpr) - \frac{R_1\eta_1 - R_0\eta_0}{BR_1}\frac{K_0(Bpr)}{K_1(BpR_1)}e^{-px_1}. \quad . \quad (20)$$

When r is small

$$pA_0(p)K_0(Bpr) = -pA_0(p)[\log(\tfrac{1}{2}Bpr) + \gamma] \quad . \quad . \quad . \quad (21)$$

approximately, γ being Euler's constant. Since $\log p + \gamma$ is the operational representation of $-\log x$, the interpretation of the last expression on the body, when $r = R(x)$, is

$$\frac{1}{2\pi}\left\{ S''(x)\log[\tfrac{1}{2}BR(x)] - \frac{\partial}{\partial x}\int_0^x \log(x-y)S''(y)\,dy \right\} \quad . \quad . \quad (22)$$

In order to evaluate the second term in equation (20) on the body, consider the function defined by

$$U\left(\frac{x-x_1}{BR_1}\right) = \frac{K_0(BpR_1)}{K_1(BpR_1)}e^{-px_1}. \quad . \quad . \quad . \quad . \quad (23)$$

This function was used by Lighthill[1], who showed that for $(x-x_1)/(BR_1)$ large, an asymptotic approximation is

$$U\left(\frac{x-x_1}{BR_1}\right) \sim \frac{BR_1}{x-x_1} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

A graph of $U(z)$ is given in Fig. 1.

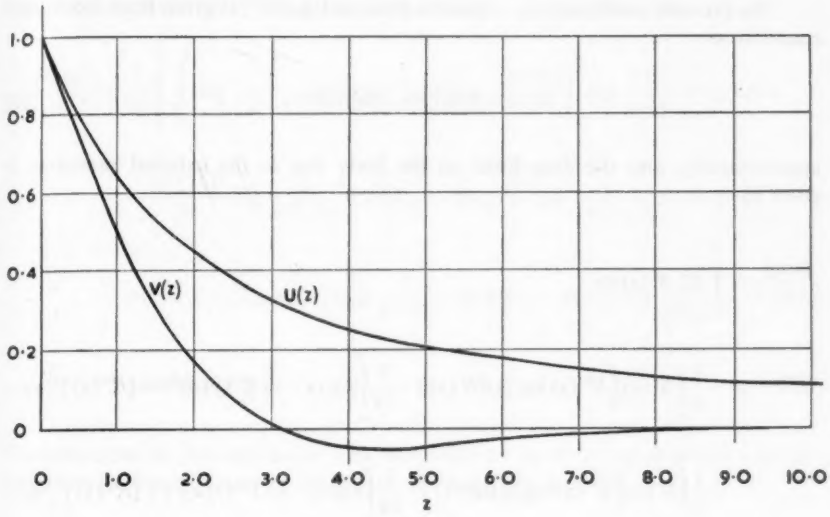

Fig. 1.

The functions $U(z)$ and $V(z)$.

When $x-x_1$ is small, $R(x) \simeq R_1$: hence, on the body,

$$\frac{K_0(Bpr)}{K_1(BpR_1)} e^{-pz_1} = U\left(\frac{x-x_1}{BR_1}\right), \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and when $x-x_1$ is large, by expanding in powers of p, an asymptotic approximation to the interpretation of the left side of equation (25) is found to be given by

$$BR_1/(x-x_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Thus $U\left(\frac{x-x_1}{BR_1}\right)$ is a good approximation to the interpretation of $K_0(Bpr)e^{-pz_1}/[K_1(BpR_1)]$ *on the body*, and so the interpretation of equation (20) on the body is approximately

$$\frac{\partial \phi_0}{\partial x} = \frac{1}{2\pi}\left\{S''(x)\log[\tfrac{1}{2}BR(x)] - \frac{\partial}{\partial x}\int_0^\pi \log(x-y)\,S''(y)\,dy\right\} - \frac{R_1\eta_1 - R_0\eta_0}{BR_1}U\left(\frac{x-x_1}{BR_1}\right)$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The pressure coefficient C_p, $=$(excess pressure)$/\frac{1}{2}\rho\mathscr{U}^2$, is given from Bernoulli's equation as

$$C_p = -2\partial\phi_0/\partial x - (\partial\phi_0/\partial r)^2 \quad . \quad . \quad . \quad . \quad . \quad (28)$$

approximately, and the drag force on the body due to the external pressures is given by

$$\frac{Drag}{\frac{1}{2}\rho\mathscr{U}^2} = \int_0^1 C_p\, S'(x)\,dx$$

$$= -\frac{1}{\pi}\int_0^{x_1-0} S'(x)\left\{S''(x)\log[\tfrac{1}{2}BR(x)] - \frac{\partial}{\partial x}\int_0^x \log(x-y)\,S''(y)\,dy + \pi[R'(x)]^2\right\}dx -$$

$$-\frac{1}{\pi}\int_{x_1+0}^1 S'(x)\left\{S''(x)\log[\tfrac{1}{2}BR(x)] - \frac{\partial}{\partial x}\int_0^x \log(x-y)\,S''(y)\,dy + \pi[R'(x)]^2\right\}dx +$$

$$+\frac{2(R_1\eta_1 - R_0\eta_0)}{BR_1}\int_{x_1+0}^1 S'(x)\,U\left(\frac{x-x_1}{BR_1}\right)dx. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Now

$$\frac{1}{BR_1}\int_{x_1+0}^1 S'(x)\,U\left(\frac{x-x_1}{BR_1}\right)dx = \left[S'(x)\,U_1\left(\frac{x-x_1}{BR_1}\right)\right]_{x_1+0}^1 - \int_{x_1+0}^1 S''(x)\,U_1\left(\frac{x-x_1}{BR_1}\right)dx,$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where $U_1(z) = \int_0^z U(z)\,dz$.

It has been shown by Lighthill[1] that for large z

$$U_1(z) \sim \log 2z + O(z^{-2}\log z),$$

so that the right hand side of equation (30) is approximately

$$S'(1)\log\left[\frac{2(1-x_1)}{BR_1}\right] - \int_{x_1+0}^1 S''(x)\log\left[\frac{2(x-x_1)}{BR_1}\right]dx. \quad . \quad . \quad (31)$$

Thus, from equations (29) and (31),

$$\frac{Drag}{\frac{1}{2}\rho \mathscr{U}^2} = \frac{1}{2\pi}\int_0^1\int_0^1 \log\frac{1}{|x-y|}\,S''(x)\,S''(y)\,dydx - \frac{S'(1)}{\pi}\int_0^1 \log\frac{1}{1-y}\,S''(y)\,dy +$$

$$+\frac{[S'(1)]^2}{2\pi}\log\frac{2}{BR(1)} + 2(R_1\eta_1 - R_0\eta_0)\int_0^1 \log\frac{1}{|x_1-y|}\,S''(y)\,dy -$$

$$-2(R_1\eta_1 - R_0\eta_0)\,S'(1)\log\frac{1}{1-x_1} + 2\pi(R_1\eta_1 - R_0\eta_0)^2\log\frac{2}{BR_1} +$$

$$+2\pi R_0^2\eta_0^2\log\frac{R_1}{R_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The total error in this expression can be shown to be $O(t^4\log^2 t)$ where t is the maximum thickness (diameter/length) of the body (*cf.* Ref. 1).

4. THE FLOW AT INCIDENCE

From considerations analogous to those used at the beginning of Section 3, we are led to write down ϕ_1 in the form

$$\phi_1 = \left\{ C_0(p)\,K_1(Bpr) + \frac{C_1(p)\,K_1(Bpr)}{BpK_1'(BpR_1)}e^{-px_1}\right\}\cos\theta, \quad . \quad (33)$$

so that

$$\frac{\partial\phi_1}{\partial r} = \left\{ Bp\,C_0(p)\,K_1'(Bpr) + \frac{C_1(p)\,K_1'(Bpr)}{K_1'(BpR_1)}e^{-px_1}\right\}\cos\theta. \quad . \quad (34)$$

On and near the body, r is small and the Bessel functions may be expanded in power series to give

$$r^2\partial\phi_1/\partial r = \{ -C_0(p)/Bp + C_1(p)\,R_1^2 e^{-px_1}\}\cos\theta \quad . \quad . \quad . \quad (35)$$

approximately. On the body

$$\pi r^2\partial\phi_1/\partial r = -\alpha\cos\theta\,S(x) = -\alpha\cos\theta\,\{F(x) + \pi(R_1^2 - R_0^2)H(x-x_1) +$$

$$+2\pi(R_1\eta_1 - R_0\eta_0)(x-x_1)H(x-x_1)\}, \quad . \quad (36)$$

from equations (8) and (11), and so

$$\pi C_0(p)/Bp = \alpha F(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$$C_1(p) = -\alpha\left\{\frac{R_1^2 - R_0^2}{R_1^2} + \frac{2(R_1\eta_1 - R_0\eta_0)}{R_1^2 p}\right\}. \quad . \quad . \quad (38)$$

Since $\phi_1 = 0$ when $x \leqslant 0$,

$$\frac{\partial \phi_1}{\partial x} = p\phi_1 = \left\{ p C_0(p) K_1(Bpr) - \frac{\alpha(R_1^2 - R_0^2) K_1(Bpr)}{BR_1^2 K_1'(BpR_1)} e^{-px_1} - \right.$$

$$\left. - \frac{2\alpha(R_1\eta_1 - R_0\eta_0) K_1(Bpr)}{BpR_1^2 K_1'(BpR_1)} e^{-px_1} \right\} \cos\theta \quad (39)$$

When r is small,

$$p C_0(p) K_1(Bpr) = \frac{C_0(p)}{Br} = \frac{\alpha F'(x)}{\pi r} \quad . \quad . \quad . \quad . \quad (40)$$

approximately.

In order to evaluate the second and third terms in equation (39), consider the functions defined by

$$V\left(\frac{x - x_1}{BR_1}\right) = -\frac{K_1(BpR_1)}{K_1'(BpR_1)} e^{-px_1} \; ; \; L\left(\frac{x - x_1}{BR_1}\right) = -\frac{K_1(BpR_1)}{BpR_1 K_1'(BpR_1)} e^{-px_1}, \quad (41)$$

and connected by the relation

$$L(z) = \int_0^z V(z)\, dz.$$

These functions were used by the present author[2] who showed that the asymptotic approximations for large argument are

$$V(z) \sim O(z^{-3}), \quad L(z) \sim 1 + O(z^{-2}), \quad . \quad . \quad . \quad . \quad (42)$$

and that $V(0) = 1$. A graph of $V(z)$ is given in Fig. 1.

When $x-x_1$ is small, $R(x) \simeq R_1$, and so, on the body,

$$\frac{rK_1(Bpr)}{R_1K_1'(BpR_1)}e^{-px_1}=V\left(\frac{x-x_1}{BR_1}\right), \quad . \quad . \quad . \quad . \quad . \quad (43)$$

and

$$\frac{rK_1(Bpr)}{BpR_1^2K_1'(BpR_1)}e^{-px_1}=L\left(\frac{x-x_1}{BR_1}\right). \quad . \quad . \quad . \quad (44)$$

When $x-x_1$ is large, asymptotic approximations to the interpretations of the left hand sides of equations (43) and (44) (found by expanding in powers of p) are respectively

$$O(1/(x-x_1)^3) \text{ and } \quad 1+O(1/(x-x_1)^3) \quad . \quad . \quad . \quad (45)$$

Thus the right hand sides of equations (43) and (44) are good approximations to the interpretations of the left hand sides *on the body* for all values of $x-x_1$, and from equation (39), $\partial\phi_1/\partial x$ on the body is approximately

$$\frac{\partial\phi_1}{\partial x}=-\frac{\alpha\cos\theta}{R(x)}\left\{\frac{F'(x)}{\pi}+\frac{R_1^2-R_0^2}{BR_1}V\left(\frac{x-x_1}{BR_1}\right)+2(R_1\eta_1-R_0\eta_0)L\left(\frac{x-x_1}{BR_1}\right)\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Also from equations (33), (37) and (44),

$$\frac{1}{r}\frac{\partial\phi_1}{\partial\theta}=-\frac{\alpha\sin\theta}{[R(x)]^2}\left\{\frac{F(x)}{\pi}+(R_1^2-R_0^2)L\left(\frac{x-x_1}{BR_1}\right)+2(R_1\eta_1-R_0\eta_0)BR_1L_1\left(\frac{x-x_1}{BR_1}\right)\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

where

$$L_1(z)=\int_0^z L(z)\,dz\sim z+O(1/z) \quad . \quad . \quad . \quad . \quad (48)$$

when z is large. From the asymptotic approximations to L and L_1, equation (47) gives

$$\frac{1}{r}\frac{\partial\phi_1}{\partial\theta}=-\alpha\sin\theta, \quad . \quad . \quad . \quad . \quad . \quad (49)$$

except over a distance $O(R_1)$ behind $x=x_1$.

The force N normal to the body axis and in the direction $\theta=0$, due to extra pressures from $\partial\phi_1/\partial x$, is given by

$$\frac{N}{\frac{1}{2}\rho\mathscr{U}^2}=\int_0^1\int_0^{2\pi}-2\frac{\partial\phi_1}{\partial x}R(x)\cos\theta\,dxd\theta$$

$$=2\alpha\left\{\int_0^1 F'(x)\,dx+\frac{\pi(R_1^2-R_0^2)}{BR_1}\int_{x_1}^1 V\left(\frac{x-x_1}{BR_1}\right)dx+2\pi(R_1\eta_1-R_0\eta_0)\int_{x_1}^1 L\left(\frac{x-x_1}{BR_1}\right)dx\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

and, by virtue of the asymptotic behaviour of L and L_1,

$$\frac{N}{\frac{1}{2}\rho\mathscr{U}^2}=2\alpha S(1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

approximately. If the lift force is measured normal to the stream direction and in the plane $\theta=0$, the lift coefficient is

$$C_L=\frac{Lift}{\frac{1}{2}\rho\mathscr{U}^2S(1)}=2\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

The moment M_0 of the lifting forces about the pointed nose of the body, tending to decrease the incidence, is given by

$$\frac{M_0}{\frac{1}{2}\rho\mathscr{U}^2}=\int_0^1\int_0^{2\pi}-2\frac{\partial\phi_1}{\partial x}R(x)\,x\cos\theta\,dxd\theta, \quad . \quad . \quad . \quad . \quad (53)$$

from which the moment coefficient is

$$C_{M0}=\frac{M_0}{\frac{1}{2}\rho\mathscr{U}^2S(1)}=2\alpha\left(1-\frac{S_m}{S(1)}\right), \quad . \quad . \quad . \quad . \quad (54)$$

S_m being the mean area of cross-section of the body.

The two results (52) and (54) are the same as for a body of revolution without an intake (*cf.* Ref. 1) and the errors can be shown to be $O(t^3\log^2 t)$ in both cases.

An additional axial force is experienced by the body at incidence, due to the

contribution to the pressure from $\partial\phi_1/r\partial\theta$; this is

$$\tfrac{1}{2}\rho\mathscr{U}^2\int_0^1\int_0^{2\pi}\left\{\alpha^2-\left(-\alpha\sin\theta+\frac{1}{r}\frac{\partial\phi_1}{\partial\theta}\right)^2\right\}\frac{S'(x)}{2\pi}\,dxd\theta$$

$$=-\tfrac{1}{2}\rho\mathscr{U}^2\alpha^2\int_0^1 S'(x)\,dx \text{ approximately}$$

$$=-\tfrac{1}{2}\rho\mathscr{U}^2\alpha^2\{S(1)-\pi(R_1^2-R_0^2)\} \quad . \quad . \quad (55)$$

The drag at incidence, measured parallel to the stream direction, contains a term $\rho\mathscr{U}^2\alpha^2$ from the component of the normal force N, in addition to the force given by equation (55), so that, if C_{D_0} is the drag coefficient at zero incidence, the drag coefficient at incidence α is

$$C_D=C_{D_0}+\{1+\pi(R_1^2-R_0^2)/S(1)\}\alpha^2, \quad . \quad . \quad . \quad . \quad (56)$$

and the error is $O(t^4\log^2 t)$.

5. THE INTERNAL LIFT FORCES

From the Prandtl-Glauert equation for subsonic flow, it may be verified that any asymmetrical disturbances at the ends of an axially symmetrical tube die away rapidly on proceeding into the tube, so that if the flow inside a propulsive duct is mainly subsonic and if no asymmetrical disturbance occurs near the exit nozzle, all the cross-momentum entering the duct will be destroyed inside. If the boundary layer on the fore-body is neglected, the rate of intake of momentum in the direction $\theta=0$ is

$$\left[\rho\mathscr{U}\int_{R_0}^{R_1}\int_0^{2\pi}(v_r\cos\theta-v_\theta\sin\theta)r\,dr\,d\theta\right]_{x=x_1-0} \quad . \quad . \quad . \quad (57)$$

By using the results for $\partial\phi_1/\partial r$ and $\partial\phi_1/(r\partial\theta)$, this may be shown to equal

$$\rho\mathscr{U}^2\alpha\pi(R_1^2-R_0^2), \quad . \quad . \quad . \quad . \quad . \quad (58)$$

which is the value of the internal force normal to the axis, and hence of the lift force to the same order of approximation. The line of action of this force cannot

be found without a detailed knowledge of the internal flow, but it may be expected to be at a distance $O(R_1)$ downstream from $x = x_1$ under most conditions of working for which the present theory applies.

The calculation of the axial force due to the internal pressures depends upon the internal conditions, including combustion, and is outside the scope of this paper.

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THE STEADY CIRCULATORY FLOW ABOUT A CIRCULAR CYLINDER WITH UNIFORMLY DISTRIBUTED SUCTION AT THE SURFACE

by

J. H. PRESTON, M.A., Ph.D., A.F.R.Ae.S.

(Aeronautics Laboratory, Cambridge University)

SUMMARY AND CONCLUSIONS

Exact solutions for the steady circulatory flow about a circular cylinder with suction applied at its surface have been obtained for the following cases:—

(*a*) the cylinder at rest and a circulation at infinity,

(*b*) the cylinder rotating and zero circulation at infinity,

(*c*) the cylinder rotating in either direction with a circulation at infinity,

provided that, in general, the suction velocity is greater than a certain limiting value.

There is a particular case of (*c*) when the circulation at the cylinder equals that at infinity in magnitude and direction, for which the circulation is constant at all radii, for all values of R—presumably for blowing as well as suction.

The addition of a uniform stream parallel to the cylinder axis does not affect the circulatory motion, and the solution for the combined flow, which is exact, has some practical interest in that it demonstrates that the boundary layer on a long, circular, rotating boss can be kept thin and that the circulatory motion outside the boundary layer can be prevented by the suitable application of suction before starting the rotation.

The flow (*a*) is important in connection with the withdrawal of the Thwaites flap. The existence of steady state solutions, coupled with the arguments of Section 5.2, make it appear highly probable that a wing employing this device will retain its lift indefinitely, when the flap is withdrawn.

For high rates of suction flow, the flows exhibit the characteristics of boundary layer flows, in that the vorticity is confined to a narrow annulus enveloping the

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cylinder and the velocity distribution near the wall tends to the well-known asymptotic distribution of Griffith and Meredith.

It would be interesting to investigate these flows experimentally and to examine the stability of the laminar flow under suction conditions.

The analysis throughout is simple and expressions are obtained for the velocity distribution, the stresses, the vorticity and the torque, and so on.

1. INTRODUCTION

The solution for the steady flow set up by a rotating cylinder in a viscous fluid is well known. The corresponding solution, when uniformly distributed suction is applied at the boundary, does not appear to have been given—at least in English literature. This problem has an academic interest, although probably not much practical significance.

There is a second problem, in which the cylinder is fixed and a circulatory flow is assumed to take place about it. This flow could be generated by rotating the cylinder and then stopping it suddenly. Without suction, there is no solution for steady flow, the motion dying away through the action of viscosity, which causes the vorticity to diffuse outwards from the surface. If suction is applied, the question arises as to whether a circulatory flow can be maintained indefinitely. The answer to this is important in connection with the problem of the withdrawal of the Thwaites flap.[1]

Briefly, the Thwaites flap is a device for obtaining high lift coefficients at small incidences, on wings with rounded trailing edges. Distributed suction is applied to the surface of the wing to withdraw effectively the whole boundary layer. A thin, short flap, capable of being set normal to the surface and of being moved round the surface, is placed at the rear of the wing. The purpose of the flap is to generate "starting" eddies and thus induce a circulation about the wing, such that the stagnation streamline leaves the flap smoothly when a steady state has been reached. Movement of the flap alters the position of the rear stagnation streamline and hence alters the value of the circulation. The flap thus coincides with the beginning of the rear stagnation streamline and has no load on it and could therefore be withdrawn into the wing. If the circulation does not die away, then it will remain constant independently of changes of incidence and speed. Thus, as Thwaites[1] has pointed out, such a wing would experience practically no change in lift on meeting a vertical gust, since $L=\rho V_0 K$. No experiments as yet have been made to see whether the circulation is maintained when the flap is withdrawn and so its value, as an anti-gust device, remains a matter for speculation.

Assuming the circulation to remain unchanged when the flap is withdrawn, a further interesting situation arises when the speed is reduced to zero. Then, there would be left a circulatory flow only, which possibly might be maintained indefinitely so long as suction is applied. This is the second flow problem previously mentioned and so the Thwaites flap provides an alternative method of generating the circulatory flow. If solutions for steady flow with circulation and with suction applied at the boundary can be found for the cylinder at rest, then it is probable that a similar state of affairs will exist when the cylinder has a forward velocity and hence the success of the Thwaites flap as an anti-gust device would be assured.

Exact solutions for the steady state have been obtained for:—

(*a*) the cylinder fixed and a circulation at infinity,

(*b*) the cylinder rotating with no circulation at infinity,

(*c*) the cylinder rotating in either direction and a circulation at infinity,

provided that the suction flow is above a certain limiting value. The flows have the characteristics of boundary layer flows in that the vorticity is effectively confined to an annulus surrounding the boundary, the width depending on the magnitude of the suction flow.

An exact solution can also be obtained when any of the above flows is combined with a uniform stream parallel to the cylinder axis. It is found that the addition of the uniform stream has no effect on the circulatory and radial components of the velocity.

Notation

Suffixes ∞ and a refer to conditions at infinity and at $r=a$ respectively.

r, θ	polar co-ordinates
t	time
q	circumferential component of velocity in direction of θ increasing
w	radial component of velocity in direction of r increasing
a	radius of cylinder
w_0	*suction* velocity at surface
p	pressure
ρ	density
μ	viscosity
ν	μ/ρ—the kinematic viscosity

ζ	vorticity
R	suction Reynolds number $=w_0 a/\nu$
p_{rr}, $p_{\theta\theta}$	normal stress components
$p_{r\theta}$	tangential stress components
T	torque on cylinder
P	ideal pump power for suction
K	circulation
q_0	$K_\infty/2\pi a =$ " potential flow " velocity at $r=a$
C	a parameter $=K_\infty/K_a$
m	a number
n	$r-a$
U	velocity at edge of boundary layer
V_0	velocity at infinity in direction normal to cylinder axis
L	lift
F_c	flux of vorticity by convection into a fixed circuit
F_d	flux of vorticity by diffusion into a fixed circuit
x	distance measured parallel to the cylinder axis
u	velocity measured parallel to the cylinder axis
U_0	velocity at infinity in direction parallel to cylinder axis
A_1, A_2, A_3, A_4 B_1, B_2, B_3, B_4	constants
R_1, R_2	particular values of R
K_1, K_2	particular values of K

See Section 4.2

2. THEORY

2.1. *Derivation of the Equation for the Velocity and its Solution. The Stresses, etc.*

The two-dimensional equations of motion are[2]

$$\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial r}+\frac{q}{r}\frac{\partial w}{\partial \theta}-\frac{q^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}-\frac{\nu}{r}\frac{\partial \zeta}{\partial \theta}, \quad . \quad . \quad . \quad (1)$$

$$\frac{\partial q}{\partial t}+w\frac{\partial q}{\partial r}+\frac{q}{r}\frac{\partial q}{\partial \theta}+\frac{qw}{r}=-\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta}+\nu\frac{\partial \zeta}{\partial r}, \quad . \quad . \quad (2)$$

where ζ, the vorticity, is given by

$$\zeta = \frac{\partial q}{\partial r} + \frac{q}{r} - \frac{1}{r}\frac{\partial w}{\partial \theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The equation of continuity is

$$\frac{\partial w}{\partial r} + \frac{w}{r} + \frac{1}{r}\frac{\partial q}{\partial \theta} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the motion is steady and independent of θ the equations (1) - (4) become

$$w\frac{dw}{dr} - \frac{q^2}{r} = -\frac{1}{\rho}\frac{dp}{dr}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$w\frac{dq}{dr} + \frac{qw}{r} = \nu\frac{d\zeta}{dr}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\zeta = \frac{dq}{dr} + \frac{q}{r}, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\frac{\partial w}{\partial r} + \frac{w}{r} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Equation (8) gives

$$wr = \text{constant},$$

$$= -w_0 a, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where a is the radius of the cylinder and w_0 is the suction velocity at the surface. Substituting from equations (8) and (9), equations (5) and (6) become

$$\frac{(w_0 a)^2}{r^3} + \frac{q^2}{r} = \frac{1}{\rho}\frac{dp}{dr}, \quad . \quad . \quad . \quad . \quad (10)$$

$$-w_0 a\left(\frac{1}{r}\frac{dq}{dr} + \frac{q}{r^2}\right) = \nu\left(\frac{d^2q}{dr^2} + \frac{1}{r}\frac{dq}{dr} - \frac{q}{r^2}\right). \quad . \quad . \quad (11)$$

q is found from equation (11) and then p is obtained from equation (10).

Equation (11) can be written

$$r^2\frac{d^2q}{dr^2}+(R+1)r\frac{dq}{dr}+(R-1)q=0, \quad . \quad . \quad . \quad . \quad (12)$$

where

$$R=\frac{w_0a}{\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

is the Reynolds number based on the suction velocity w_0.

Equation (12) is a second order homogeneous equation and the solution is of the type[3]

$$q=r^m. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Substituting (14) in (12) we obtain

$$m(m-1)+(R+1)m+(R-1)=0,$$

or

$$m^2+Rm+(R-1)=0. \quad . \quad . \quad . \quad . \quad (15)$$

The roots of this equation are

$$m_1=-1,$$

$$m_2=1-R, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and the solution of equation (12) is

$$q=A_1r^{m_1}+A_2r^{m_2}$$

Substituting from (16)

$$q=A_1r^{-1}+A_2r^{1-R}, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

provided that $m_2\neq m_1$, *i.e.* $R\neq 2$.

If $R=2$, $m_2=m_1=-1$, the solution is[3]

$$q=\left\{B_1r^{m_1}+B_2\frac{\partial}{\partial m_1}(r^{m_1})\right\}_{(m_1=-1)}$$

$$=r^{-1}(B_1+B_2\log_e r). \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The Stresses

The mean normal stress $(-p)$ is obtained from equation (10) by substitution for q from equations (17) or (18) and then integrating. The stress components are[2]

$$p_{rr} = -p + 2\mu \frac{\partial w}{\partial r} = -p + 2\mu \frac{dw}{dr}, \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$p_{\theta\theta} = -p + 2\mu \left(\frac{1}{r} \frac{\partial q}{\partial \theta} + \frac{w}{r} \right) = -p + 2\mu \frac{w}{r}, \quad . \quad . \quad . \quad (20)$$

$$p_{r\theta} = \mu \left(\frac{\partial q}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{q}{r} \right) = \mu \left(\frac{dq}{dr} - \frac{q}{r} \right), \quad . \quad . \quad . \quad (21)$$

which are readily computed.

The torque T on the cylinder is given by

$$T = 2\pi a \, (p_{r\theta})_{r=a} \times a. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The ideal power required by the suction pump (efficiency = 1.0) neglecting the resistance of the porous surface is

$$P = (p_\infty - p_a - \tfrac{1}{2}\rho w_0^2) \times 2\pi a w_0. \quad . \quad . \quad . \quad (23)$$

2.2. *Solution for Cylinder Fixed, Finite Circulation at Infinity*

The Velocity Distribution

The boundary conditions are

$$\left.\begin{aligned} q &= 0, \quad r = a \\ K\,(=2\pi q r) &\longrightarrow K_\infty, \quad r \longrightarrow \infty \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

For $R > 2$

For the first condition, putting $q = 0$, $r = a$ in (17), we obtain

$$A_2 = -A_1 a^{R-2}. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Hence

$$q = A_1 (r^{-1} - a^{R-2} r^{1-R}). \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Put

$$A_1 = q_0 a, \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where $$q_0 = \frac{K_\infty}{2\pi a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Then $$\frac{q}{q_0} = \frac{a}{r} - \left(\frac{a}{r}\right)^{R-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and $$\frac{K}{K_\infty} = \frac{qr}{q_0 a} = 1 - \left(\frac{a}{r}\right)^{R-2}, \quad . \quad . \quad . \quad . \quad . \quad (30)$$

so that the second condition is satisfied.

For $R < 2$ there is no solution satisfying the last condition of (24).

For $R = 2$ equation (18) must be used and is rewritten

$$\frac{q}{q_0} = \frac{a}{r}\left(B_1 + B_2 \log_e \frac{r}{a}\right). \quad . \quad . \quad . \quad . \quad (31)$$

At $r = a$, $q = 0$, hence $B_1 = 0$,

and $$\frac{K}{K_\infty} = \frac{qr}{q_0 a} = B_2 \log_e \left(\frac{r}{a}\right). \quad . \quad . \quad . \quad . \quad (32)$$

At $r \longrightarrow \infty$, $K \longrightarrow K_\infty$, and at $r/a \longrightarrow \infty$, $B_2 \log (r/a) = 1.0$.

This makes $B_2 = 0$ and hence $q/q_0 = 0$ for all values of r/a, and $K/K_\infty = 0$ for all values of r/a other than $r/a = \infty$. These are the limiting values obtained from equations (29) and (30), $R \longrightarrow 2.0$.

The Pressure

From equation (10)

$$\frac{1}{\rho}\frac{dp}{d(r/a)} = w_0^2 (a/r)^3 + q^2 a/r. \quad . \quad . \quad . \quad . \quad (33)$$

But from equation (29)

$$q^2 = q_0^2 \left\{\left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)^R + \left(\frac{a}{r}\right)^{2(R-1)}\right\}, \quad . \quad . \quad . \quad . \quad (34)$$

hence $$\frac{1}{\rho}\frac{dp}{d(r/a)} = (w_0^2 + q_0^2)\left(\frac{a}{r}\right)^3 + q_0^2 \left\{\left(\frac{a}{r}\right)^{2R-1} - 2\left(\frac{a}{r}\right)^{R+1}\right\}.$$

Integrating

$$\frac{p}{\frac{1}{2}\rho q_0^2} = -\left\{1+\left(\frac{w_0}{q_0}\right)^2\right\}\left(\frac{a}{r}\right)^2 - \frac{1}{R-1}\left(\frac{a}{r}\right)^{2(R-1)} + \frac{4}{R}\left(\frac{a}{r}\right)^R + \text{constant.}$$

Now $r/a \longrightarrow \infty$, $p \longrightarrow p_\infty$, say.

Hence

$$\frac{p_\infty - p}{\frac{1}{2}\rho q_0^2} = \left(\frac{a}{r}\right)^2\left[1+\left(\frac{w_0}{q_0}\right)^2 + \frac{1}{R-1}\left(\frac{a}{r}\right)^{2(R-2)} - \frac{4}{R}\left(\frac{a}{r}\right)^{R-2}\right]. \quad . \quad (35)$$

and at $r=a$,

$$\frac{p_\infty - p_a}{\frac{1}{2}\rho q_0^2} = 1+\left(\frac{w_0}{q_0}\right)^2 + \frac{1}{R-1} - \frac{4}{R} \quad . \quad . \quad . \quad . \quad (36)$$

The Power

The ideal power required by the pump is, from equation (23),

$$\frac{P}{\frac{1}{2}\rho q_0^3 a} = 2\pi\left(\frac{w_0}{q_0}\right)\left[1+\frac{1}{R-1}-\frac{4}{R}\right]. \quad . \quad . \quad . \quad (37)$$

The Stresses

Substitute the expression for w (equation (9)) in equations (19) and (20) to obtain

$$\frac{p_{rr}}{\frac{1}{2}\rho q_0^2} = -\frac{p}{\frac{1}{2}\rho q_0^2} + \frac{4}{R}\left(\frac{w_0}{q_0}\right)^2\left(\frac{a}{r}\right)^2, \quad . \quad . \quad . \quad (38)$$

$$\frac{p_{\theta\theta}}{\frac{1}{2}\rho q_0^2} = -\frac{p}{\frac{1}{2}\rho q_0^2} - \frac{4}{R}\left(\frac{w_0}{q_0}\right)^2\left(\frac{a}{r}\right)^2, \quad . \quad . \quad . \quad (39)$$

where p is given by equation (35).

In certain circumstances, *e.g.* with q_0 small, $r/a \longrightarrow 1.0$, $R \simeq 2$ and w_0/q_0 large, p_{rr} and $p_{\theta\theta}$ will differ appreciably from $(-p)$ the mean of the normal stresses.

The tangential stress $p_{r\theta}$ is obtained from equation (21), which may be written

$$p_{r\theta} = \mu\frac{q_0}{a}\left(\frac{d\,(q/q_0)}{d\,(r/a)} - \frac{q/q_0}{r/a}\right).$$

Whence from equation (29)

$$(p_{r\theta})_{r=a} = \mu\frac{q_0}{a}(R-2). \quad . \quad . \quad . \quad . \quad . \quad (40)$$

The Torque

The torque on the cylinder, from equation (22), is given by

$$\left.\begin{aligned} T &= 2\pi\mu a q_0 (R-2) \\ &= \mu K_\infty (R-2) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

2.3. *Solution for Cylinder Rotating, Zero Circulation at Infinity*

If, in equation (17), we put

$$A_1 = 0, \quad A_2 = q_a a^{R-1},$$

then

$$\frac{q}{q_a} = \left(\frac{a}{r}\right)^{R-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

This gives

$$\left.\begin{aligned} q &= q_a, \ r = a \\ q &= 0, \ r = \infty \\ K &= 0, \ r = \infty \end{aligned}\right\} \quad R > 2$$

where q_a is the peripheral velocity.

Calculations of the pressure and normal stresses follow the same lines as already developed for the cylinder at rest and will not be given here. There is an interest in computing the tangential stress at the surface as it is required in the torque computation.

Writing equation (21) as

$$(p_{r\theta})_{r=a} = \mu \frac{q_a}{a}\left(\frac{d(q/q_0)}{d(r/a)} - \frac{q/q_0}{r/a}\right)_{r=a},$$

$$= \mu \frac{q_a}{a} R. \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Whence from (22), the torque is found to be

$$T = 2\pi\mu q_a a R. \quad . \quad . \quad . \quad . \quad . \quad (44)$$

2.4. *Solution for Cylinder Rotating, Arbitrary Circulation at Infinity*

I am indebted to M. B. Glauert for pointing out the possibility of an arbitrary circulation at infinity, which is not related to the peripheral speed q_a.

To obtain the velocity distribution ($R > 2$) write (17) as

$$q = A_1\left(\frac{a}{r}\right) + A_2\left(\frac{a}{r}\right)^{R-1},$$

and since at $r = a$, $q = q_a$,

$$q_a = A_1 + A_2,$$

or
$$\frac{q}{q_a} = \frac{A_1}{q_a}\left(\frac{a}{r}\right) + \left(1 - \frac{A_1}{q_a}\right)\left(\frac{a}{r}\right)^{R-1},$$

or
$$\frac{q}{q_a} = C\left(\frac{a}{r}\right) + (1-C)\left(\frac{a}{r}\right)^{R-1}, \quad . \quad . \quad . \quad . \quad . \quad (45)$$

and
$$\frac{K}{K_a} = \frac{2\pi qr}{2\pi q_a a} = \frac{qr}{q_a a} = C + (1-C)\left(\frac{a}{r}\right)^{R-2}, \quad . \quad . \quad (46)$$

whence, for $R > 2$,
$$\frac{K_\infty}{K_a} = C. \quad . \quad . \quad . \quad . \quad . \quad . \quad (46a)$$

Thus the constant C represents the ratio of the circulation at infinity to that at the surface. By giving C the appropriate value in equation (45) the results of Sections 2.2 and 2.3 can be reproduced.

If $C = 1$

$$\frac{q}{q_a} = \left(\frac{a}{r}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

for all values of R and hence of the suction velocity w_0.

$C = 0$ reproduces the flow with the circulation at infinity zero, dealt with in Section 2.2.

$C = \pm\infty$ is equivalent to the cylinder at rest, if we put

$$Cq_a = q_0.$$

Expressions for the pressure and stresses are readily obtained, but will not be given here. The torque is of more interest and substituting q/q_a from equation (45) in equations (21) and (22) gives for the torque

$$T=-2\pi\mu q_a a\{2C+(1-C)R\} \quad . \quad . \quad . \quad . \quad . \quad (48)$$

If, in this equation, we put $C=0$, we obtain the result given in Section 2.3 equation (44), and if we put $C=\infty$ and $Cq_a=q_0$, we obtain equation (41) of Section 2.2. $C=1$ gives the well-known expression for the torque on a rotating cylinder in a viscous fluid and suction does not affect this.

The vorticity is also of considerable interest. Equation (7) may be written

$$\zeta=\frac{q_a}{a}\left\{\frac{d(q/q_a)}{d(r/a)}+\frac{q/q_a}{r/a}\right\},$$

whence, substituting from equation (45), we obtain

$$\zeta=\frac{q_a}{a}(C-1)(R-2)\left(\frac{a}{r}\right)^R. \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Hence, for *large values of* R, the vorticity is contained in a comparatively narrow annulus surrounding the cylinder, or in other words, a boundary layer exists.

2.5. *Asymptotic Conditions, R large*

As a consequence of this, the differential equation for q, equation (12)

$$\frac{r^2d^2q}{dr^2}+(R+1)r\frac{dq}{dr}+(R-1)q=0,$$

can be simplified.

We put $$r=a+n, \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

where n is the distance measured from the surface, along any radius. r^2 and r, where these occur explicitly, can be replaced by a^2 and a respectively. The term $(R-1)q$ is negligible compared with the remaining terms, if R is large. Also $(R+1)\simeq R$ and the equation becomes, since $R=w_0a/\nu$,

$$\frac{d^2q}{dn^2}+\frac{w_0}{\nu}\frac{dq}{dn}=0. \quad . \quad . \quad . \quad . \quad . \quad (51)$$

The solution is

$$q = A_3 \exp(-w_0 n/\nu) + B_3, \quad . \quad . \quad . \quad . \quad . \quad (52)$$

which is the asymptotic velocity distribution and is identical with that found by Griffith and Meredith for the flow past an infinite plate, through which uniform suction is applied.

The constants A_3 and B_3 are to be determined from the conditions:

$$q = q_a, \quad n = 0,$$

$$q \longrightarrow U, \quad n \text{ large},$$

where U is the velocity just outside the boundary layer and is found from the external potential flow.

2.6. *Addition of a Uniform Stream in the Direction of the Cylinder Axis*

The circulatory flows considered may be combined with a uniform stream parallel to the cylinder axis. The details of the solution are given in the Appendix. The solution is still exact and the circulatory component of the motion is identical with that already discussed, being unaffected by the motion parallel to the axis. The axial motion is independent of x, the distance along the axis, and of θ, the angular co-ordinate, but it is a function of r. The axial component of velocity as a function of r tends to the well-known Griffiths and Meredith distribution, equation (52), when R is large.

3. RESULTS

Typical velocity distributions are shown in Figs. 1 and 2. Fig. 1 shows results for the cylinder at rest with a circulation at infinity, for various values of the suction Reynolds number, $R = w_0 a/\nu$. All the curves for $R > 2$ approach that for $R = \infty$ asymptotically. For small values of R the approach is slow, but for large values of R it is rapid. The velocity distributions have a maximum close to the cylinder before the asymptotic approach to the $R = \infty$ curve sets in. For large R, the velocity rises extremely rapidly close to the surface and the asymptotic approach to the $R = \infty$ curve is short. Two such distributions are shown on enlarged scales of r/a in Figs. 1 (*b*) and 1 (*c*), the latter being indistinguishable from the asymptotic distribution given by the equation (52), with appropriate values for the constants.

Figure 2 shows three different types of flow with the cylinder rotating. For (*a*), the circulation at the cylinder equals that at infinity and one velocity distribution

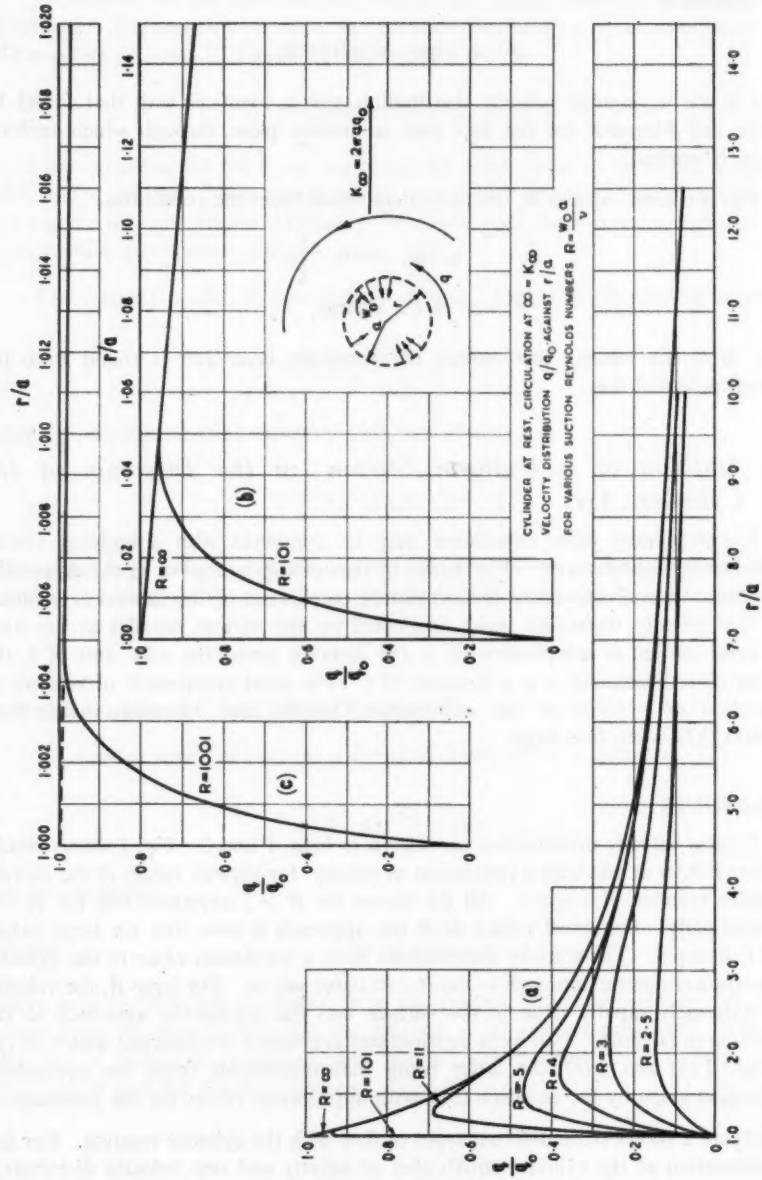
CYLINDER AT REST, CIRCULATION AT ∞ = K∞
VELOCITY DISTRIBUTION q/q0 AGAINST r/a
FOR VARIOUS SUCTION REYNOLDS NUMBERS R = w0 a / ν
K∞ = 2πq0a
(a)
(b)
(c)
R=∞
R=101
R=11
R=5
R=4
R=3
R=2.5
R=1001
r/a
q/q0

Fig. 1.

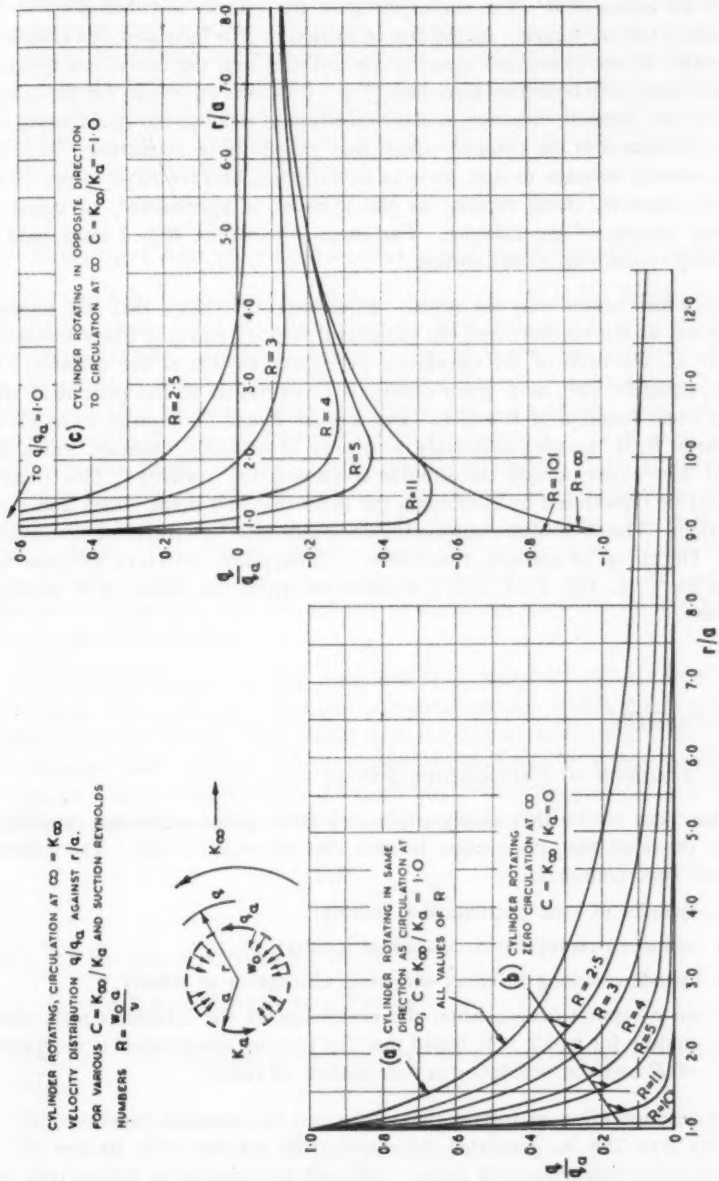
CYLINDER ROTATING, CIRCULATION AT ∞ = K∞
VELOCITY DISTRIBUTION q/qa AGAINST r/a.
FOR VARIOUS C = K∞/Ka AND SUCTION REYNOLDS NUMBERS R = w0a/ν
(a) CYLINDER ROTATING IN SAME DIRECTION AS CIRCULATION AT ∞, C = K∞/Ka = 1·0 ALL VALUES OF R
(b) CYLINDER ROTATING ZERO CIRCULATION AT ∞ C = K∞/Ka = 0
(c) CYLINDER ROTATING IN OPPOSITE DIRECTION TO CIRCULATION AT ∞ C = K∞/Ka = -1·0
R=2·5
R=3
R=4
R=5
R=11
R=101
R=∞
q/qa
r/a
K∞
Ka
w0
q
qa

Fig. 2.

holds for all values of R. The curves (b) show the velocity distributions when the circulation at infinity is zero. As the rate of suction or R is increased, the circulatory flow remains at rest closer and closer to the cylinder and the curves are typical of those associated with boundary layer flow. Fig. 2 (c) shows the results for the cylinder rotating in the opposite direction to the circulation at infinity, the speed being such that the circulation at the cylinder equals that at infinity in magnitude. It is seen that the velocity reverses in sign, as is to be expected, and for large values of R it is mainly negative, rising rapidly, as the cylinder is approached, to equal the peripheral velocity of the cylinder. For cases (b) and (c), $R > 2$ must hold for the boundary conditions to be satisfied.

Two other results may be noted. Equation (48) shows that the torque is proportional to the viscosity and the peripheral velocity and is a linear function of R and of C, the ratio of the circulation at infinity to that at the cylinder. The vorticity, equation (49), at a given radius, is proportional to the peripheral speed and is a linear function of R and C. For a given R and C, it varies as $(a/r)^R$ and so for large R, it becomes effectively zero very close to the cylinder, and a near potential flow exists outside the annulus containing the vorticity. This potential flow could be reproduced by combining the flows due to a point vortex and a sink at the origin. The streamlines outside the boundary layer are therefore equi-angular spirals. The curves of vorticity distribution, ζ/ζ_a (equation (49)) take the same form as those for q/q_a, Fig. 2 (b), with 1.0 subtracted from the values of R shown on the curves.

4. DISCUSSION

4.1. Generation of Circulatory Flows

It has been shown that steady circulatory flows about a circular cylinder are possible, provided that the suction is such that $R = w_0 a/\nu > 2.0$. The following types have been considered: :—

(a) cylinder at rest, circulation at infinity;

(b) cylinder rotating, no circulation at infinity;

(c) cylinder rotating in either direction, circulation at infinity;

(d) an exception occurs when the circulation at the cylinder equals that at infinity for which it is found that the velocity distribution is independent of R and the circulation is independent of radius.

The question arises as to how these flows can be generated experimentally. A circulatory flow can be generated by rotating the cylinder with suction off—the circulation being independent of radius—although presumably an infinite time must

elapse before the vorticity can diffuse to infinity and give rise to a circulation there. Application of suction then corresponds to type (*d*). Type (*a*) can be reproduced by stopping the cylinder and keeping suction on. Type (*c*) occurs if the speed and/or direction of rotation is altered with suction on. To generate type (*b*), suction, such that $R > 2$, is applied to the cylinder at rest and the cylinder is then rotated. An alternative method of generating (*a*), (*c*) and (*d*) is to use the Thwaites flap as described in the introduction—Section 1.

4.2. *Note on Change of Circulation for Flows Variable with Time*

The solutions computed in Section 2 have been those for steady conditions with the requirement that $R > 2$, in general. Now the method of generating these flows experimentally, as already described, brings in the time element. Consider type (*a*) above, in which the cylinder is at rest and there is a circulation at infinity. Suppose we have attained steady conditions for a particular value of suction corresponding to R_1 and we then reduce the suction to R_2. Then, even assuming that finally the steady velocity distribution corresponding to R_2 is attained, the circulation at infinity is arbitrary and need not necessarily correspond to the initial value. During the change from one steady state to the other, the instantaneous velocity distributions will be different from the steady types, in the range of $R_1 > R > R_2$. Now for the steady state, provided $R >> 2$, we can always determine a radius such that the vorticity is effectively zero, and we assume this to be so for the unsteady flow.

In order to discover whether there will be a change of circulation at infinity, use is made of a well-known theorem in hydrodynamics, which relates the rate of change of circulation in a fixed circuit with the flux of vorticity into the circuit[2]. This theorem states that the rate of change of circulation in a fixed circuit equals the rate at which vorticity enters the circuit by convection plus the rate at which vorticity enters the circuit by diffusion. Or symbolically

$$\frac{dK}{dt} = F_c + F_d, \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

where F_c = the flux of vorticity by convection,

F_d = the flux of vorticity by diffusion.

If we add $\partial q/\partial t$ to equation (6) and multiply by $2\pi r$ and put $wr = -w_0 a$, from (9), we obtain

$$\left(\frac{dK}{dt}\right)_r = 2\pi w_0 a\zeta + 2\pi\nu r \frac{\partial \zeta}{\partial r}, \quad . \quad . \quad . \quad . \quad (54)$$

where the first term represents F_c and the second F_d, both in the direction of r decreasing.

Now for $r=a$, $q=0$ and $K_a=0$ for all times. Hence the net flux of vorticity $(F_c+F_d)_a$ across $r=a$ is zero for all times, since $dK_a/dt=0$. If we take a circuit of sufficiently large radius, such that $F_c \to 0$, $F_d \to 0$, then even though $F_c \neq -F_d$, $(F_d+F_c) \to 0$, and hence for this circuit $dK/dt=0$, *i.e.* $K=$constant, and hence has the value before suction is decreased. For a circuit of small radius a reduction of circulation occurs between the initial and final states. Hence $dK/dt<0$, and $F_c+F_d<0$, *i.e.* the vorticity diffuses more rapidly *outwards*, than it is convected inwards and there is a net flow of vorticity *out* of the circuit given by

$$-\int_0^\infty (F_c+F_d)\,dt=K_2-K_1.$$

This is evident from equation (49) for the vorticity, where a reduction of R modifies its distribution and increases the width of the region in which it is contained. In the steady states, before and after the reduction of suction, $F_d=-F_c$ everywhere, so that an exact balance is maintained between the flux of vorticity by convection inwards and the flux by diffusion outwards. When $R<2$ no steady state solution exists, other than that for which there is no circulatory motion. Thus, if the suction falls below that for $R=2$, convection by suction is unable to arrest the outward diffusion of vorticity and the circulation everywhere will fall to zero.

It seems highly probable that once a circulation at infinity has been established under suction conditions, then it will remain unchanged by changes of suction and/or direction and magnitude of rotation of the cylinder, provided that the suction is such that $R>2$ and provided the field of flow is infinite*. If the field of flow is finite, then vorticity will diffuse inwards from the outer boundary and the circulation will fall gradually to zero.

Withdrawal of the Thwaites flap should not lead to ultimate loss of circulation when the cylinder has a forward velocity or when the wind is on in a wind tunnel; for although the field of flow is limited in the latter case by the tunnel walls, the vorticity generated there is confined to the boundary layer at the walls and therefore cannot reach the cylinder.

* The arguments, although plausible, do not constitute a proof that the circulation will remain unchanged during changes of magnitude and direction of rotation and/or suction. The transient state associated with these changes will be investigated in a later paper.

APPENDIX

Addition of a Uniform Stream in the Direction of the Cylinder Axis

Denote the cylinder axis by x and let the velocity be u in this direction, with a value U_0 at $r=\infty$. Then, if q, w, u and p are independent of x and θ, the equations of motion in the notation of this report reduce[4] to equations (9), (10), (11) and the following equation,

$$w\frac{du}{dr}=\nu\left\{\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}\right\} \quad . \quad . \quad . \quad . \quad . \quad (55)$$

Thus the equations governing the circulatory flow remain unchanged by the addition of a flow parallel to the axis of the cylinder and hence the solution already given stands.

Substitute for w from equation (9) in equation (55) to obtain

$$\left.\begin{aligned}\frac{d^2u}{dr^2}+(R+1)\frac{1}{r}\frac{du}{dr}&=0\\ R&=\frac{w_0a}{\nu}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (56)$$

The solution of this is $u=A_4r^{-R}+B_4$

or

$$\frac{u}{U_0}=A_4\left(\frac{r}{a}\right)^{-R}=B_4. \quad . \quad . \quad . \quad . \quad . \quad (57)$$

The boundary conditions are

$$\left.\begin{aligned}\frac{r}{a}\rightarrow\infty,\ \frac{u}{U_0}&\rightarrow 1\\ \frac{r}{a}=1.0,\ \frac{u}{U_0}&=0\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (58)$$

whence

$$\left.\begin{aligned}B_4&=1\\ A_4&=-1\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (59)$$

and

$$\frac{u}{U_0}=1-\left(\frac{a}{r}\right)^{-R}, \quad . \quad . \quad . \quad . \quad (60)$$

which, for R large, becomes identical with the asymptotic exponential distribution (equation (52)).

The combined flow given here has some practical interest in that it demonstrates that the boundary layer on a long, circular, rotating boss can be kept thin and the circulatory motion outside the boundary layer can be prevented by the suitable application of suction.

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NOTE ON H. TEMPLETON'S PAPER "CONTROL REVERSAL EFFECTS ON SWEPT-BACK WINGS"

by

W. H. WITTRICK, M.A.

(Department of Aeronautical Engineering, The University of Sydney)

Mr. Templeton's paper "Control Reversal Effects on Swept-back Wings" (page 3, *The Aeronautical Quarterly*, Volume I, Part I) contains in Section 2 of the Appendix an argument which does not appear to be applicable to any but the simplest type of wing, namely one in which all the flexural and torsional rigidity is given by a single spar.

Consider a swept-back *stressed skin* wing in which the ribs are *chordwise* (*i.e.* parallel to the flight path), and suppose further that the ribs are closely spaced and rigid in their own planes. This appears to be the basic assumption made in the Appendix of Mr. Templeton's paper. It is then clear that, if the wing is distorted by aerodynamic loads, the only wing sections which will remain undistorted are the *chordwise* sections. In particular, a *normal* section (*i.e.* normal to the spar), will undergo a change of shape. To speak of the rotation of a normal section seems incorrect because this immediately raises the question of how to define the rotation of something which does not retain its initial shape. The concept of a rotation automatically presupposes a rigid body movement. From this it follows that the term GJ, the torsional rigidity of a normal section, has no meaning.

Although the *form* of equation (1) of Mr. Templeton's paper appears sound, the expressions which he gives for K_t and K_m in terms of the torsional and flexural rigidities of a normal section appear to be incorrect for a stressed skin wing.

A further result of the argument above is that the conventional definition of a flexural axis for an unswept wing cannot apply to a swept wing of this type. The conventional definition is surely only a special case of a more general one which

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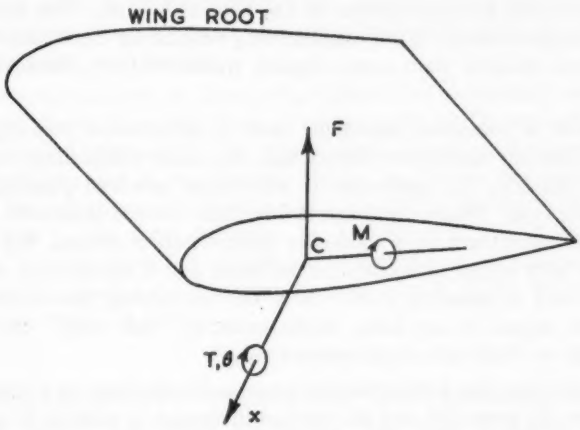
embraces all angles of sweep. This is seen in the accompanying sketch, which shows a *chordwise* section of a swept-back wing and the bending and torsion loads acting at the section. The torque T acts about an axis perpendicular to the section, while the bending moment M acts about an axis in the section. The rotation of the section is θ in the same sense as the torque T. Then, if C is the *flexural centre* of the section, the twist $d\theta/dx$ depends only on T and M, *and not on the shear force* F. The *flexural axis* is merely the locus of the flexural centres, and, using Mr.



Templeton's notation, the relation between $d\theta/dx$ and the bending moment and torque may be expressed as

$$\frac{d\theta}{dx} = K_t T + K_m M.$$

In the special case of an unswept wing K_m becomes zero and the twist then depends only on the torque about the flexural axis. The outer portion of a swept-back wing with ribs in the normal direction is structurally no different from an unswept wing, and the conventional definition of the flexural axis applies.

A further point needs clarification. Consider two swept-back wings which are identical in all respects, except that the ribs are in the normal direction in the first and chordwise in the second. Then the flexural axis of the first (defined in the conventional way) does not necessarily coincide with the flexural axis of the second (defined as above).

This note is based on unpublished numerical work, using a theory which is exact within the limits of the assumptions of closely spaced rigid chordwise ribs.

REVIEWS

Distribution of Deformation (a new method of structural analysis), C. V. Kloucek. (Translated from the Czech and German editions by A. H. Waddell-Zalud and F. H. Zalud), Práce, Prague, 1949, 510 pp.

This work is an English translation, by A. H. Waddell-Zalud and F. H. Zalud, from the Czech and German editions of Dr. Kloucek's work. The translation is a liberal one, certain sections of the original being omitted for clarity and brevity, and is much more readable than many English translations of continental books of this sort.

This book is concerned essentially with a deformation counterpart of the well-known moment distribution theory and, like most publications on the latter, is presented primarily for application to stiff-jointed unbraced plane frames of the building frame type. Where the usual Hardy-Cross method deals with moments at joints and estimates them by a successive approximation process, the author here deals mainly with angles of rotation at the joints and a special way of estimating them. The work is presented in two parts, the first dealing with structures whose joints are not subject to any linear displacement or "side sway," and the second with structures in which such displacements occur.

In the first part, the basic relations between the rotations at a given joint and its neighbours are presented and the method illustrated in relation to a continuous beam, and to a multi-storey plane frame with three floors and four columns. This first part appears to have been published originally in 1940, and is based on various earlier papers by the author, dating from 1938.

The second part of the book deals with the problem of estimating the effects of sidesway. Both single storey and multi-storey frames are considered and the deformation method is illustrated by numerical examples in each case. Sidesway due either to wind loading or unsymmetrical vertical loading is studied. Further examples include a two-column skyscraper frame and a Vierendeel truss.

While the deformation distribution method is presented in a very full and generally satisfactory manner in this publication, an English reader is presented at the outset with a difficulty that it is not easy to resolve. This is the author's claim in his sub-title and throughout the work that the method discussed is a new one based upon his own work, starting in 1937. Now the fact that the various well-known moment distribution processes had joint rotation and displacement counterparts was, the writer thinks, appreciated by a number of English research workers before the recent World War. In America the possibility of such deformation methods was probably realised rather earlier, a paper by L. E. Grinter on one such method being published in the "Proceedings of the American Society

of Civil Engineers" in 1936. In his text book "Theory of Modern Steel Structures," published in 1937, Grinter devotes a chapter to the "Analysis of continuous frames by balancing angle changes," which is based upon his earlier paper.

This is not to suggest that Dr. Kloucek's method does not represent an independent and in some ways original development. Thus he observes and neatly utilises the geometric series nature of the successive approximations envisaged, a matter that in England has been little used except for the analysis of a plane frame loaded transversely to its plane, by D. Williams and J. Taylor. He has developed an interesting process for dealing with sidesway—commonly numerically troublesome —by replacing the multi-storey frame under its actual lateral loading by an equivalent cantilever under the same loading.

A number of arguments in favour of a deformation distribution method rather than a moment distribution method are of course noted by Dr. Kloucek. Of these, the reviewer is most impressed by the greater degree of self-checking possible in the deformation process and by its physical clarity. As Grinter has remarked in his text book, "the procedure of balancing angle changes is susceptible to expression in many forms"; the form favoured by Dr. Kloucek, his "Distribution of Deformation," is in this book fully set out and well supported by numerical examples. This is primarily a book for a specialist designer of stiff jointed frames, and is to be welcomed as such.—A.G.P.

Computation curves for compressible fluid problems, C. L. Dailey and F. C. Wood. John Wiley & Sons Inc., New York, Chapman & Hall Ltd., London, 1949. 33 pages. 26 diagrams. 12/- net.

This is a useful compilation of data, dealing with problems which occur in compressible flow involving discontinuities, commonly called shock waves. The book contains no information that has not been known to aerodynamicists interested in this subject, but the presentation is specially adapted to assist its practical application.

The material is sub-divided into four sections. The first gives the theoretical relations and formulae used. The three following sections give graphs presenting the state of a gas as a function of the local Mach number and the effect on its properties caused by heat addition in a constant area channel, the relations for flow across a plane oblique shock wave, and those for flow past a cone.

It is well known that in non-isentropic flow a general solution of the differential equations defining the state of flow for given boundary conditions can only be obtained in certain simple cases. Linearisation of these equations enables mathematical solutions to be obtained, and a huge volume of work has been and is being done on the basis of this method, the physical significance of which is still largely undetermined. The solutions presented in the present volume are not based

on a linearisation of the fundamental differential equations; it is however somewhat dangerous to describe them therefore as "exact theory," as the authors do. They are, of course, exact in the sense that any purely mathematical conclusions from a given set of assumptions can be called exact. The validity of such assumptions as applied to any physical process remains to be proved however. In this respect, the authors are not as explicit and consistent as seems desirable. For example, in the case of supersonic flow around a cone, the assumptions that the shock wave generators are straight lines and that pressure on the cone surface is constant are suddenly introduced in the middle of the argument by reference to "observed facts." Since observations of the case treated, *i.e.* on flow around an infinite smooth cone, can hardly have been available, the reference is presumably to the conical fronts of bodies such as projectiles. The question immediately arises whether this is true for all configurations at all Mach numbers at which shock waves are formed? Since such questions are not discussed in the present publication, it would have been better to state all assumptions made at the beginning of each section and, where possible, describe the physical conditions to which they may be applied.

In the light of these remarks, it is amusing to quote the authors' concluding remarks to the application of the Taylor-Maccoll theory to the calculation of flow between the conical shock and the cone surface:

"It was assumed above, as a basis for the method of numerical integration that a solution existed and that it was regular. It cannot, therefore, be argued that its existence is proved by the fact that it has been calculated. However, it can be asserted that the solution which has been calculated is empirically observed to be the same as that which occurs in nature, and therefore exists and is regular"!—E.P.

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